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# THEORETICAL MECHANICS.

*AN ELEMENTARY TREATISE.*

BY

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# PREFACE.

In preparing the present work, which was designed to include in a single volume of moderate compass the elementary portions of Theoretical Mechanics, no formal division of the subject into Kinematics, Statics and Kinetics has been made. The topics often included under the first head it was thought best to introduce separately, each at the point where it is required for immediate application to the treatment of the motions produced by forces. For example, the expressions for radial and transverse accelerations are not introduced until required in the discussion of Central Forces.

The subject of Statics is, to be sure, to a large extent separable from the idea of motion. But, on the one hand, as has been recognized in all recent treatises, the fundamental notions of force are best presented, and the Parallelogram of Forces is best established, on the basis of the Laws of Motion. This requires what may be called a dynamical introduction to Statics. On the other hand, the subject cannot be completed without the Method of Virtual Velocities, an application of the Principle of Work. This principle, which is dynamical as involving forces acting through spaces, advantageously precedes the study of kinetics into which time enters explicitly, and prepares the student for the notion of Kinetic Energy, or work embodied in motion.

Accordingly, in the present volume, Chapter I consists of such a kinetical introduction to the whole subject as is referred to

above; Chapters II–VI are purely statical; Chapter VII treats of the dynamical Principle of Work with its application to Statics and to the notion of the Potential Function; and the remaining chapters treat of purely kinetical topics.

The chapters are further subdivided into sections followed by copious lists of graded examples aggregating over 500 in number; many of these were taken from examination papers set at the Naval Academy, and not a few were prepared expressly for this work.

In these examples, as well as in the numerical illustrations introduced in the text, gravitation units of force have for the most part been employed. These units in fact not only have the advantage of being rendered familiar to us by the common usages of every-day life, but they are actually more convenient than absolute units in mechanical problems, since in them the forces arise principally from the weights of bodies. Thus their use is forced upon even those writers who most deprecate the employment of a variable unit of force. The conception of an absolute unit of force, dependent upon mass and motion and not upon weight, is indeed essential to the gaining of correct ideas of the nature of force. Hence the introduction by Prof. James Thomson of the poundal, which serves this purpose when the English system of weights and measures is used, has been of very great value. At the same time the employment of the pound as a unit of mass as well as a unit of force has been the cause of confusion, so that a student is sometimes in doubt whether the result of the use of a formula is the number of pounds or of poundals, or, as he may phrase it, whether the formula is expressed in gravitation or in absolute units. To prevent this confusion, care has been taken in the present volume, while using gravitation units, to avoid such expressions as, for example, "a mass of 6 pounds," and to speak instead of "a body whose weight is 6 pounds." There is no doubt that the same body would be intended in either expression, but the former would imply that, in the formula $W = mg$, 6 is the numerical value of $m$, and the latter that 6 is the numerical value

of $W$. Inasmuch as the pound, though an absolute "standard" of mass, is properly called and legally styled a "unit of weight," the latter is the more natural course. Accordingly the student is directed on page 13 to follow it and to remember that all the forces are thus expressed in local pounds. If the result is desired in poundals, neither is the formula changed nor is the result found in one unit and then changed to the other, but the number of pounds is taken as the numerical value of $m$.

For the same reason, we should not say that the "weight" of a body varies when it is taken to a place where $g$ has a different value, because the number which legally expresses its weight remains the same. The force of its gravity has indeed changed, but it is (when we use gravitation units) the unit of this force, and not its numerical measure, which has changed.

In the treatment of kinetics, the conception of the forces of inertia has been freely employed, and that without the apologies that some writers have thought necessary. It would seem that the resistance of a body in motion to acceleration in any direction is as much entitled to be regarded as a force as is the resistance of other bodies which, in the case of a body at rest, prevent motion. By including the latter as forces, we obtain the idea of a system of forces in equilibrium; so also, by including the former as forces, we extend this idea to the case of a body in motion, and D'Alembert's Principle presents itself in the form of "kinetic equilibrium," instead of requiring for its statement a set of hypothetical "effective forces."

The study of Mechanics is here supposed to follow an adequate course in the Differential and Integral Calculus, and to form a very important application of its principles. But, when these applications occur, the results are not merely presented in the shape of general formulæ in the notation of the Calculus, leaving the student unaided in the process of evaluation. Instead of this, pains has been taken to instruct the student in the methods best adapted in various cases to obtaining numerical results. Particularly in the treatment of statical moments and of moments

of inertia it is hoped that the book will be found a useful supplement to the course of instruction in the processes of integration.

Throughout, the practice of relying upon substitution in general formulæ is discouraged as far as possible, and the opposite practice inculcated—namely, that of applying general principles directly to the problem in hand.

Special prominence is given to those results which it is the most important to make familiar to the student of Applied Mechanics, and to the readiest ways of recalling them when they have slipped the memory.

Although preference is given to analytical processes, a not inconsiderable use is made of graphical methods. These have, however, been introduced rather as diagrammatic aids to the comprehension of general principles, and to the calculation of numerical results, than as methods of obtaining results by measurement from accurately constructed diagrams—the latter belonging rather to the province of Applied Mechanics.

W. W. J.

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# THEORETICAL MECHANICS.

## CHAPTER I.

## DEFINITIONS AND LAWS OF MOTION.

### I.

### Motion in a Straight Line.

1. Mechanics is the science which treats of the motions of *material bodies*, and the causes of these motions.

A *force* is an action, applied to a material body or to any part of it, which when unresisted produces motion. A *solid* body is one which resists relative motion between its parts, so that it does not readily change its shape. When the forces under consideration can produce no change of shape, the body is said to be *rigid*, and it moves only as a whole. If the motion of a rigid body is such that every straight line drawn in its substance remains always parallel to its original position, the motion is said to be one of *translation*. When this is the case, it is obvious that the motion of a single point of the body (whether it be in a straight or in a curved line) is sufficient to determine completely the motion of the body.

The whole amount of matter contained in a body is often imagined to be concentrated at a single point. In this case it is called a *material particle*. The motion of a body in translation is completely represented by the motion of a particle.

2. We discuss in this book only the motions of rigid bodies, and at first consider motions of translation, so that the body may be regarded as a particle, and the forces as acting at a single point. In this first chapter, we consider the general relations between forces and the motions they produce, from which is derived the mode in which they are measured and subjected to mathematical analysis.

## Velocity or Speed.

3. When a body is in motion, we have to consider both the *speed* and the direction of the motion. The term *velocity* is often used to include both these notions; in such case, the velocity of a body is not said to be constant unless the direction of the motion as well as its speed is unchanged; that is, unless the motion is rectilinear as well as uniform.

In the first section of this chapter, we shall suppose the motion to be in a single straight line, so that speed only will at present be considered.

The speed is *uniform* when the spaces described in any intervals of time are always proportional to the intervals. When this is the case, its measure is the number of units of space described in a unit of time. Thus, if $t$ denotes the number (integral or fractional) of units of time in any interval, and $s$ denotes the number of units of space described or passed over in that interval, the velocity is uniform when the ratio of $s$ to $t$ is the same for all corresponding values of $s$ and $t$. Now putting $v$ for this constant ratio, we have

$$v = \frac{s}{t}. \qquad (1)$$

In this equation $v$ is the value of $s$ corresponding to $t = 1$, and we take this as *the numerical measure of the velocity.* It is necessary to specify the units of time and space employed; thus we speak of a speed of 10 feet per second, of 15 miles per hour, of a mile in two minutes and ten seconds, and so forth.

4. By means of equation (1), we can obtain the numerical measure of a constant speed from any given corresponding values

of the space and time, and thus pass from one set of units to another. For example, to express the velocity of 30 miles per hour in feet per second. Here 30 miles is given as the space corresponding to the time one hour; expressing the values of $s$ and $t$ in equation (1) in feet and seconds respectively, we have

$$v = \frac{30 \times 5280\text{f}}{60 \times 60\text{s}} = 44^{\text{f}}/_{\text{s}}.$$

The arithmetical work shows that 44 ft. is the space corresponding to one second, and the customary mode of expressing the unit of velocity,* namely in the fractional form $^{\text{f}}/_{\text{s}}$, is suggested by the mode in which the symbols for the units of space and time occur in the equation. This result may therefore be expressed thus:

$$30^{\text{m}}/_{\text{h}} = 44^{\text{f}}/_{\text{s}};$$

and it is one which it is useful to remember, as giving the ratio between the numerical measures of any velocity as expressed in these units. We shall regard the *foot* and the *second* as the standard units of time and length, and therefore the *foot per second* as the standard unit of velocity.

## Variable Speed.

5. When the spaces passed over in equal intervals of time are not equal, the speed is variable, and the quotient arising from dividing the space by the time gives what may be called the *average speed* for the given time. But at any given instant of time the speed has a definite value of which the numerical measure is *the number of units of space which would be described in a*

* Separate names for units of velocity have been proposed, but have not been generally accepted. It is in fact better to keep the fundamental units of space and time in evidence. It is said that the "knot" is the only single term for a unit of velocity in general use: thus we speak of a speed of 12 knots, meaning 12 sea-miles per hour. But the term knot is also often used as synonymous with sea-mile.

*unit of time if the body moved uniformly throughout that interval with the speed which it had at the instant considered.*

Hence, if $s$ denote the distance of the body, at any time $t$, from some fixed origin of distances taken on the path of the particle (here supposed to be a straight line), we have, by the definition of the derivative,

$$v = \frac{ds}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This expression may also be regarded as the limiting value (when $\Delta t$ is indefinitely diminished) of the ratio

$$\frac{\Delta s}{\Delta t};$$

where $\Delta t$ is the increment of $t$, the time reckoned from some fixed instant taken as the origin of time, and $\Delta s$ is the corresponding increment of $s$, that is, the space passed over in the interval $\Delta t$. (See Art. 390, Diff. Calc.) Writing equation (2) in the form

$$ds = vdt,$$

we see that, when the value of $v$ is known for every instant or value of $t$ (in other words, when $v$ is given as a function of $t$), $s$ is given by the equation

$$s = \int vdt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which involves a constant of integration depending upon the position of the body at some given time. Again, using limits, we may write for the space described in a given interval

$$s - s_0 = \int_{t_0}^{t} vdt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $s_0$ and $s$ correspond respectively to $t_0$ and $t$, the values of the time at the beginning and end of the interval in question.

It is to be noticed that the result of supposing $v$ *constant*, and making $s_0$ and $t_0$ each equal to zero, is $s = vt$, equivalent to equation (1), Art. 3.

6. The simplest example of a variable velocity, expressed as a known function of the time, is that of a body falling freely from a position of rest. It has been shown by experiment that the velocity at the end of any time after the instant when the body was dropped is proportional to the time; so that we may put

$$v = gt,$$

where $g$ is a constant. This equation implies that $v = 0$ when $t = 0$ (that is, the body was at rest at the instant from which $t$ is reckoned), and that $v = g$ when $t = 1$, so that $g$ is the velocity of the body at the end of one unit of time. Using our standard units, it is found that this velocity is about $32^f/_s$; hence, supposing $g = 32$, the equation shows, for example, that the velocity at the end of the first half-second is $16^f/_s$, at the end of 2 seconds it is $64^f/_s$, etc.

7. If now we use this expression for $v$ in equation (3), Art. 5, and perform the integration, we shall have

$$s = \tfrac{1}{2}gt^2 + C,$$

where $C$ is the constant of integration. Now if we agree to measure the space $s$ from the position of rest, having already assumed that $t = 0$ when $v = 0$, we must have $s = 0$ when $t = 0$; therefore $C = 0$, so that $s = \frac{1}{2}gt^2$ is the space fallen through in $t$ seconds from rest. In particular, putting $t = 1$, we find 16 feet for the space fallen through in the first second; putting $t = 2$, 64 feet is the space fallen through in the first 2 seconds. The difference of these, or 48 feet, is the space fallen through during the 2^d second, as would be directly obtained by using the limits 1 and 2 in equation (4). Since this 48 feet is described during a single second it measures the *average speed*, Art. 5, for that

second. It will be noticed that, in this case, the average speed is midway between the least and the greatest speed which occur during the interval, namely, $32\,{}^{f}/_{s}$ and $64\,{}^{f}/_{s}$, which correspond respectively to the beginning and to the end of the interval.

## Acceleration and Retardation.

**8.** The motion of a body is said to be hastened or *accelerated* when the velocity is increasing, and it is said to be *uniformly accelerated* when the *increments of velocity which take plcae in any two intervals of time are proportional to the intervals.* Thus, the motion considered in Art. 6, namely, that of a freely falling body, is a case of uniformly accelerated motion; for the expression $v = gt$ shows that in any one second the velocity changes from $gt$ to $g(t + 1)$, that is, it receives the increment $g$; in any two seconds it receives the increment $2g$; in any half-second, the increment $\frac{1}{2}g$; and so on.

Under these circumstances, *the increment of velocity received in a unit of time* is taken as *the measure of the acceleration.* Thus, in the case of the falling body, the acceleration is constant and equal to $g$. Supposing the motion to start from rest at the beginning of the interval, so that $v = 0$ when $t = 0$, the acceleration is the same as the velocity acquired in the first second, or the quotient arising from dividing the velocity acquired in any interval by the number of units of time in that interval.

Suppose, for example, that a train getting under way acquires a velocity of 18 miles per hour during one minute; assuming the acceleration to be constant, what is its measure in the standard units—that is, the foot and second? The velocity $18\,{}^{m}/_{h} = 18 \times \frac{44\,{}^{f}/_{s}}{30}$ (see Art. 4); dividing this by the number of seconds in which it is acquired, and denoting acceleration by $\alpha$, we have

$$\alpha = \frac{18 \times 44\,{}^{f}/_{s}}{30 \times 60s} = \frac{44}{100}\,{}^{f}/_{s^2}.$$

Thus the foot-second unit of acceleration is a *gain* of velocity at the rate of *one foot per second per second*, and the process shows how this naturally gives rise to the symbol $^f/_{s^2}$.

9. The motion of a body is said to be *retarded* when the velocity is decreasing, and the rate of loss of velocity is called *the retardation*. For example, suppose that a stone projected along the ground with the velocity of 20 feet per second is observed to come to rest in 4 seconds. Here the velocity $20^f/_s$ is lost in 4 seconds; hence, if we suppose the rate of loss to be constant, there is a loss of $5^f/_s$ per second, that is, the retardation is $5^f/_{s^2}$.

## Variable Acceleration.

**10.** The acceleration $\alpha$ is defined as the rate of the velocity, whether that rate be constant or variable. Hence, using the notation of Art. 5, we have

$$v = \frac{ds}{dt}, \qquad \alpha = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

When we are dealing with motion in both directions along a straight line, it is necessary to assume one direction as the positive one for measuring $s$ from the origin. Then $v$ is positive when the body is moving in this direction, so as to increase a positive or numerically decrease a negative value of $s$. In like manner, $\alpha$ is positive when a positive value of $v$ is increasing or a negative value of $v$ is numerically decreasing; on the other hand, $\alpha$ is negative when a positive value of $v$ is decreasing or a negative value numerically increasing.

For example, when a heavy body is projected vertically upward, if the space is measured upward, the velocity is at first positive and decreases; hence there is a retardation. The value of $\alpha$ is therefore negative; and, on account of this *negative acceleration*, the positive velocity is lost in a certain time, and after that converted into a negative and numerically increasing velocity. As a particular case, suppose the upward velocity of

projection to be $128^f/_s$, the negative acceleration being 32. Reckoning the time $t$ from the instant of projection, the loss of velocity in $t$ seconds is $32t$; hence the velocity at any instant is given by

$$v = 128 - 32t.$$

Putting $v = 0$, we find that the whole velocity is lost when $32t = 128$, that is, when $t = 4$. Again, if we put $t = 5$, we find $v = 128 - 160 = -32$, showing that at the end of 5 seconds the body is descending, and has acquired a negative velocity of $32^f/_s$.

11. An example of motion with variable acceleration is afforded by any vibratory motion, like that of a pendulum. For, since the velocity changes sign alternately from $+$ to $-$, and from $-$ to $+$, the acceleration must also change sign. The simplest case of vibratory motion in a straight line is that which is called *harmonic*, in which the distance of the particle from the origin at the time $t$ is given by the equation

$$s = a \sin \omega t. \qquad (1)$$

When $t = 0$, the particle is at the origin; when $\omega t = \frac{1}{2}\pi$ or $t = \dfrac{\pi}{2\omega}$, it is at the distance $a$ from the origin on the positive side; when $t$ has twice this value, it is again at the origin; at the end of three times this interval, $s = -a$, the particle is at its greatest distance on the negative side; and so on. By successive differentiation, equation (1) gives

$$v = a\omega \cos \omega t. \qquad (2)$$
$$\alpha = -a\omega^2 \sin \omega t. \qquad (3)$$

A comparison of equations (3) and (1) shows that the acceleration is negative whenever $s$ is positive, and positive whenever $s$ is negative. It is zero at the origin; and at that point $v$ has its greatest positive or negative value. This is in accordance with the principles of maxima and minima, since $\alpha$ is the derivative of $v$.

## The Laws of Motion.

**12.** The science of Mechanics is based upon certain first principles which must be regarded as established by experience. These, having been first clearly formulated by Sir Isaac Newton in the *Philosophiae Naturalis Principia Mathematica*, are known as Newton's Laws of Motion. We shall in the succeeding articles give the three laws in literal translation from the Latin of the *Principia*, each followed by the necessary explanations.

## Inertia.

**13.** LAW I.—*Every body keeps in its state of rest or of moving uniformly in a straight line, except so far as it is compelled by forces acting on it to change its state.*

This law, which is sometimes called the *Law of Inertia*, asserts that, while some external cause which we call *force* is necessary to put a body in motion, no such external action is necessary to *keep* it in uniform rectilinear motion after it has acquired a velocity; but, on the contrary, force is then required either to deflect it from a rectilinear path, or to alter its speed. This is contrary to the notion of the ancients, who regarded the earth as at rest, and attributed the observed tendency of bodies put in motion to come to rest to an inherent property of matter which they called *inertia.* On the other hand, we now hold that the earth itself is in motion, but that this does not in any way disturb the relative motion of bodies with respect to it. We regard inertia as opposed to *any* change of motion; so that, when bodies already in motion come to rest relatively to the earth,* the fact must be attributed to external causes or forces.

We cannot completely prove the first law of motion experimentally, because it is impossible to free the body on which we experiment entirely from the action of external forces; but we can show that the nearer we approach to this condition the nearer we realize a state of uniform rectilinear motion.

* It is noteworthy that Galileo, who was the first to hold correct views of the nature of force and motion, maintained also that the earth was in motion.

## The Measure of Force.

14. LAW II.—*Change of motion is proportional to the moving force acting, and takes place in the straight line in which the force acts.*

This is the most important of the three laws. We defer to the next section its application to forces and motions in various directions, and here consider only the case of a single force acting upon a freely moving body. The *direction* of the force is of course that of the straight line in which the body, starting from rest, begins to move under the influence of the force *acting freely*—that is, when no other forces are acting. This line is called *the line of action* of the force. If, after the body has acquired motion in this line, the force continues to act in the same direction, the body will continue to move in the same straight line; for there is no reason why it should deviate from it to one side rather than the other. In the case of the single body now under consideration, "change of motion" means change of velocity. The second law therefore asserts that the change of velocity produced in any interval of time is proportional to the force acting during that time.

15. It follows that, if the changes of velocity in all equal intervals of time while the force is acting are equal (in other words, if the *acceleration*, Art. 8, is constant), the intensity of the force is constant. Thus, because the motion of a freely falling body, Art. 6, is found to be a case of uniformly accelerated motion, we infer that the force which urges a body downward is a constant one. It is thus independent of the velocity with which the body is moving. On the other hand, the force of the wind upon a body moving before it is not constant, but depends in part upon the velocity of the moving body, for the acceleration in this last case is not constant; in fact it disappears when the body has acquired the velocity of the wind. Again, the acceleration of a body sinking in water is not constant, because the constant force due to the body's *weight in water* is resisted by a force which depends upon the velocity. The acceleration in this case vanishes

when the resistance becomes equal to the weight in water, and the body then descends with a uniform velocity.

So far as it relates to the motion of a single body, we may therefore express the second law as follows : *Force is measured by the acceleration it produces in a freely moving body.*

## Mass.

**16.** We come now to the consideration of the action of forces upon different bodies. When the bodies are regarded as particles, the only respect in which they differ is in quantity of matter, which is called *mass.* Moreover, so far as motion of translation is concerned, the size and shape of the body, and the mode of distribution of the matter within the volume, is of no consequence. The comparison of the masses of bodies is *practically* effected by means of their *weights*, as indicated by the common balance. If two bodies are equal in weight, we assume that they are equal in mass. It will presently be shown why this assumption is correct ; but it is important to notice that the mass of a body is really measured by its *resistance to change of motion.*

For, if two equal forces act in the same direction upon two equal bodies starting from rest, the bodies will acquire the same velocity, and will move side by side. They may therefore be considered as forming a body of double mass acted upon by a double force. Thus a double force is required to produce a given acceleration in a double mass, and in like manner it can be shown in general that *the force required to produce a given acceleration is proportional to the mass moved.*

In other words, the *inertia* of a body, which, as stated in Art. 13, is its resistance to change of motion (or the quality of matter by virtue of which it requires force to produce change of motion), is proportional to the mass of the body.

## Equation of Force and Motion.

**17.** The results of Arts. 15 and 16 may be combined in the statement that force is *jointly proportional* to the mass upon which it acts freely and the acceleration it produces in that mass. This

is the form in which the proposition was stated by the older writers, who always used proportion in comparing magnitudes of different kinds. But the modern practice is to adopt units for the various magnitudes, in accordance with which, the force is said to be proportional to the *product* of the mass and the acceleration, meaning thereby the product of the numerical measures of the mass and of the acceleration. When two quantities are said to be proportional, one of them is put equal to the product of the other by a constant; but in the present case it is the universal practice *to adopt such units of force, mass and acceleration* that, denoting the numerical values of the quantities by $F$, $m$ and $f$, respectively, we shall have the equation

$$F = mf. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have seen in Art. 8 how the unit of the acceleration $\alpha$, or $f$ depends upon the units of space and time; and it is to be noticed that, by virtue of this equation, $f$ stands not only for *the acceleration,* but for *the force acting upon each unit of mass* contained in the body.

## The Units of Force and Mass.

**18.** The forces most familiar to us are the weights of bodies due to the attraction of gravity. It is found that the downward acceleration produced by gravity* at any place is the same for all bodies, irrespective of the material of which they are composed. Denoting this acceleration by $g$, and the weight by $W$, equation (1) above gives

$$W = mg. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is this equation with its constant value of $g$ that justifies the assumption, mentioned in Art. 16, that *equality of weight indicates equality of mass.*

Accordingly, the units of weight established by law, such as the *pound* and the *gramme*, serve commonly as units of mass. Thus

* In proving this experimentally, it is necessary that the experiment be tried in a vacuum, so as to remove the resistance of the air and allow gravity to act "freely."

a pound of any material means that quantity whose mass is the same as that of the standard imperial pound preserved in the Standards Office in London, or its copy in the Treasury Building in Washington.

**19.** Now the force with which gravity acts upon the mass of a pound came very naturally to be also called a pound, and it is convenient in the practical applications of mechanics to use the unit of weight as the unit of force, because the forces generally arise from the weights of bodies. It is plain, however, that in the equation $W = mg$ we cannot use the pound as the unit of $m$ as well as of $W$. Now, since we intend to use the pound as the unit of force, it must be remembered that *the number of pounds* which a body weighs is to be taken as *the numerical value of* $W$ (not of $m$). There is generally no occasion to give a numerical value to $m$; for, when $m$ appears in a formula, it may be replaced by $W/g$.

The pound and other units of force founded upon the action of gravity upon standard masses are called *gravitation units of force.*

**20.** It is found that the value of $g$ is not the same for all places upon the earth's surface, but undergoes a variation of nearly one per cent, depending principally upon the latitude of the place. It follows that the pound and other gravitation units of force are variable; and, in order to give precise information about the intensity of a force expressed in pounds, it is necessary to know also *the local value* of $g$. Thus a body which weighs 6 pounds would stretch a spring, like that of a spring-balance, slightly further at a place near the pole than it would at a place near the equator. We should not on this account say that the body *weighs* more at one place than another, for its "weight" is the number of units of weight which it balances (namely, in the illustration 6), which is independent of the place where it is weighed. This number is also the number of *local pounds* in *the force of its gravity.* The numerical measure of the force (in this case 6) remains unchanged; but the unit of force, and therefore its actual magnitude, is different at the two places.

It follows that a spring-balance, in order to indicate weight correctly, must be graduated in accordance with the local value of $g$ at the place where it is to be used.

## Absolute Units of Force.

**21.** For purposes of scientific research, in various departments of physics, it is essential to have an *absolute* unit of force independent of the value of $g$. For this purpose the mass of the national standard of weight is taken as the unit of mass. The equation $F = mf$ then shows that the unit of force is *that force which produces the acceleration unity in the standard mass.* When the pound is taken as unit of mass and the foot and second as units of space and time, the unit of force thus obtained is called the *poundal.* The poundal may also be defined as *that force which, acting for one second upon the mass of the imperial pound, will produce in it a velocity of one foot per second.*

In using this system of units, the number of pounds in the "weight" of the body is taken as the value of $m$; and then $W$ or $mg$ is the number of poundals in the force of its gravity. Hence, to reduce the numerical measure of a force given in local pounds to poundals, it is necessary to multiply it by the local value of $g$. For example, at a place where $g = 32$, the gravity of the standard pound would exert a force of 32 poundals; so that the poundal would at that place be half an ounce in gravitation measure, and at any place on the earth's surface it is not far from half an ounce.

In the system of C. G. S. units, the gramme (or mass of a cubic centimeter of distilled water at 4° C.) being the unit of mass, and the centimeter the unit of length, the corresponding unit of force is called the *dyne.* The dyne is therefore *that force which, acting for one second upon a gramme, will give it a velocity of one centimeter per second.*

## Momentum and Impulse.

**22.** The product, $mv$, of the mass of a body and its velocity is called its *momentum.* When the velocity varies we have for the rate of change of momentum, since $m$ is constant,

$$\frac{d(mv)}{dt} = m\frac{dv}{dt} = m\alpha; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

that is, *the rate of momentum is the product of the mass and the acceleration.* We have seen in Art. 17 that this product measures the intensity of the force acting, which we have denoted by $F$; so that the equation $F = m\alpha$ is by equation (1) equivalent to

$$Fdt = mdv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The whole action of the force in the interval of time $t$ is the integral of this expression between the limits 0 and $t$, corresponding respectively to the beginning and the end of the interval. Thus

$$\int_0^t Fdt = mv - mv_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $v_0$ and $v$ are the velocities at the beginning and end of the interval. The first member of equation (3) is called *the impulse* of the force in the time $t$, and the equation expresses that *the impulse is measured by the whole change of momentum produced by it.**

**23.** When $F$ is constant the expression for the impulse takes the simpler form $Ft$, the product of time and force. If, moreover, there is no initial velocity, $v_0 = 0$, and equation (3) takes the simple form

$$Ft = mv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The numerical measure of an impulse may be expressed either in absolute or in gravitation units. Suppose, for example, that a force acting for 10 seconds is observed to give a body weighing 6 pounds a velocity of $5^f/_s$. The *absolute* value of the impulse is here 30, as found by putting $m = 6$ and $v = 5$ in equation (4);

* This appears to be the more accurate expression of Newton's second law in modern phraseology, for the "moving force acting" (*vis motrix impressa*) as used by Newton corresponds to what is now called impulse; whereas the term *force* is now applied only to pressure or intensity of force.

hence, putting $t = 10$, the force, supposed constant during the interval, is 3 poundals. But if we wish the result in gravitation units we must, in accordance with Art. 19, put $\frac{W}{g}$ for $m$ in equation (4), and the result is $10F = \frac{6}{g} \times 5$, whence $F = \frac{3}{g}$ is the value of the force in local pounds. The force, in this illustration, is an absolute one; hence its *measure in local pounds* depends upon $g$. At a place where $g = 32$ its measure is $1\frac{1}{2}$ ounces.

When $t = 1$, the impulse and the force have the same numerical value, and equation (4) then corresponds to the second mode of expressing the measure of force given in Art. 21. But it must be remembered that *impulse* corresponds to *momentum*, and force to the rate of momentum or *mass-acceleration.*

## Reaction.

**24.** Law III.—*There is always a reaction opposite and equal to an action, or the actions of two bodies upon one another are always equal and oppositely directed.*

This third law of motion, which is often called the *law of reaction*, assumes that every force acting upon a body and tending to produce motion is of the nature of a tendency in the body to approach or to recede from some other body. This tendency is called the *action* of the second body upon the first. The law of reaction asserts that in every case there is an equal force acting upon the second body, which is called the *reaction* of the first body upon it. Moreover, these actions take place in two opposite directions along the straight line joining the two bodies, which are here considered as particles.

When the mutual action takes place *at a distance* there is said to be an *attraction* or a *repulsion* between the bodies according as they tend to approach or to recede. For example, the weight of a body is due to an attraction between the body and the earth: an electrical action may be an attraction or a repulsion. If, in the case of action at a distance, the bodies are free to move, the equal forces acting simultaneously on the two bodies give rise to

equal impulses, so that, by the second law, the momenta produced in any given interval are equal. It follows that, if the bodies start from rest, the velocities are inversely proportional to the masses.*

**25.** When a body acted upon by forces is in contact with another body, and thereby prevented from moving, the action of this second body on the first is called a *resistance*. For example, when a heavy body rests upon a table, its weight may be regarded as a downward force acting upon the table. By the law of reaction, the table exerts an equal upward force upon the body. The body is here treated as a particle, so that there is a single point of contact, the vertical line through which is the common line of action of the two forces. So in more complex cases, wherever solid bodies are in contact, there may exist an equal action and reaction in some line passing through the point of contact.

These forces may be called *passive* forces, in distinction from those which are capable of acting at a distance. They are only called into being by the *active* forces, and thus cannot act "freely" so as to produce motion.

**26.** When a solid body by virtue of *cohesion* resists forces tending to separate its parts, the parts may be regarded as two bodies between which a mutual action and reaction exist. For example, when a weight is suspended by a rod from a support, the part of the rod below any given point acts with a downward force upon the part above the point, and the part above acts with an equal upward force upon that below. The rod is, in this case, said to be in a state of *tension*.

Again, if the rod be interposed between the weight and a support below it, the part of the rod above any given point acts downward upon the part below, and there is an equal upward reaction of the part below upon that above. The rod is, in this case, said to be in a state of *compression*.

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* Of course, in the case of a falling body, the mass of the earth is so great, relatively, that its motion may be ignored and the relative motion attributed entirely to the falling body.

## Transmission of Force.

**27.** In the illustrations given above, the downward force of the weight of the body is said to be *transmitted* by the rod to the point of support, which may thus be any point in the line of action of the force. Thus, by supposing the force to act upon a rigid body, *its point of application may be transferred to any point in the line of action*; and, according to the mode of transference, the force will appear at its new point of application as a *pull* or a *thrust* acting upon some new body. This principle is known as *the transmissibility of force.*

A *flexible body*, like a string, which does not resist change of shape but which does resist change of length, may also be used to transmit force in the form of a pull; in other words, it may be in a state of tension, but not in one of compression.

EXAMPLES. I.

1. What is the numerical value of a velocity of 22 feet per second when the units of space and time are the mile and hour? 15 miles per hour.

2. A sprinter makes a 100-yard dash in $9\frac{3}{4}$ seconds. What is his average velocity in feet per second? $30\frac{10}{13}$.

3. A mile run is made in $4^m\ 24^s$. What is the average rate in feet per second? 20.

4. A railway train travels 100 miles in 2 hours. Find the average velocity in feet per second. $73\frac{1}{3}$.

5. Two bodies start together from the same point and move uniformly along the same straight line in the same direction; one body moves at the rate of 15 miles per hour, and the other body at the rate of 18 feet per second. Determine the distance between them at the end of a minute. 240 ft.

6. If the bodies move with the velocities of the preceding example but in *opposite* directions, at the end of what time will they be 200 feet apart? 5 sec.

7. A body starts from a point and moves uniformly along a straight line at the rate of 30 miles per hour. At the end of half a minute another body starts from the same point in the same

direction, and moves uniformly at the rate of 55 feet per second. Find the time and distance the second body must travel to overtake the first. 2 min.; 6600 ft.

8. A steamer takes $a$ minutes to run a measured sea-mile with the tide, and $b$ minutes to return against the tide. Determine the speed through the water. $30\dfrac{a+b}{ab}$ knots.

9. A railway train, whose full speed is 60 miles an hour, is 20 seconds in getting under full headway with uniform acceleration from rest. What is the value of the acceleration when the units are the foot and the second? $4\frac{2}{5}$.

10. When under headway, how far will the train in Ex. 9 be behind the position it would have had if it had been under full headway at the time and place of starting? $\frac{1}{6}$ of a mile.

11. A train moving 35 miles per hour is brought to rest by the action of its brakes in 10 seconds; what is the retardation in foot-second units? $5\frac{2}{15}$.

12. A 100-pound shot is acted upon by a force which in one second produces a velocity of 100 feet per second. What is the measure of the force in pounds at a place where $g = 32.2$ feet? 310.56 pounds.

13. A spring-balance is adjusted at a place where $g = 32.2$; what is the true weight of a body which by this balance appears to weigh 10 pounds at a place where $g = 32$? 10 lbs. 1 oz.

14. If a force of 8 poundals acts for 10 seconds, what is the impulse in gravitation units? And if the body acted upon weighs a pound, what is the velocity produced? $\dfrac{80}{g}$; $80^{f}/_{s}$.

15. If two bodies starting from rest attract one another, prove that they will meet at a point dividing their distance in the inverse ratio of their masses.

16. Two trains, 250 and 440 feet long respectively, pass each other on parallel tracks with equal velocities in opposite directions. A passenger in the shorter observes that it takes the longer exactly 4 seconds to pass him. What is the velocity? $37\frac{1}{2}{}^{m}/_{h}$.

## II.

## Composition of Motions.

**28.** The motion of a particle from the position $A$ to the position $B$ is called *the displacement* $AB$; it is most simply represented by a straight line drawn from $A$ to $B$, and thus has a definite length and a definite direction. In the diagrams the displacement $AB$ may be distinguished from the opposite displacement $BA$ by an arrow-head. In a motion of translation (see Art. 1), every point of the solid body is regarded as undergoing the same displacement; in other words, parallel displacements of the same length and direction are regarded as identical. A line thus regarded as representing a translation is called a *vector*; and accordingly a vector is considered as having *only* the qualities of length and direction, and not any particular position in space. The term vector is also applied to lines which represent other conceptions which involve only direction and magnitude.

**29.** Two consecutive displacements, such as $AB$ and $BD$ in Fig. 1, are equivalent to a single displacement $AD$. In this *composition of displacements*, it is to be noticed that the order of composition is immaterial; for, if we complete the parallelogram $ABDC$, the vectors $AC$ and $CD$ are identical with $BD$ and $AB$ respectively, and being compounded in the reverse order lead to the same result $AD$. Thus, the composition of vectors, like ordinary addition, is a *commutative* operation; that is, one in which the parts may be interchanged without affecting the result. The operation is in fact sometimes called *geometrical addition*, and the vector $AD$ is called the sum of the vectors $AB$ and $AC$.

## Composition of Velocities.

**30.** If we suppose the displacement of a particle to take place by uniform rectilinear motion, and *in one unit of time*, the vector which represents the displacement represents also the velocity, both in amount and direction. We may further suppose the

two displacements $AB$ and $AC$ (Fig. 1) to take place uniformly and simultaneously. This is most clearly conceived of by supposing one of the motions, say $AB$, to be that of an extended body like a ship, while the other, $AC$, is the motion of a body *relatively* to the ship, that is, as it appears to a person standing upon the deck. Then, if these motions, with the velocities $AB$ and $AC$, take place simultaneously, the body, which at the end of one second has moved to the point $C$ relatively to the ship, will on account of the motion of the ship be found at $D$. Again, at the end of any fractional part of the second, while the ship has moved through the distance $AE$ the body will have moved, relatively to it, through $EF$, which bears the same ratio to $AC$ that $AE$ does to $AB$; therefore, by the principle of similar triangles, it will be on the diagonal $AD$, at a distance $AF$ from $A$, which bears the same ratio to $AD$. Thus the point will describe the line $AD$ with uniform velocity, and $AD$ represents the actual velocity in space, or, as it is called, *the resultant velocity*, both in magnitude and direction.

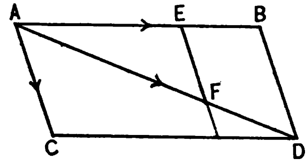

FIG. 1.

31. The above construction, as applied to velocities, is sometimes called *the parallelogram of velocities*; but it is to be noticed that we need only to construct the triangle $ABD$, in other words, that velocities are combined, like displacements, by geometrical addition of the representing vectors. The resultant speed, which is the length of the third side of the triangle, and the direction of the resultant motion may then be found by the solution of a plane triangle of which the sides and included angle are given. In particular, the resultant speed cannot be greater than the sum, or less than the difference, of the given speeds.

32. If the resultant and one component velocity are given, the process of finding the other component is one of *subtraction* of vectors. For example, suppose it is required to row from the point $A$ on one bank of a river to the point $B$ on the opposite bank in a given time; the speed of the current being known, in

what direction and with what speed *relatively to the water* must the boat be rowed? In this case, the resultant direction $AB$ and the resultant speed are given. Let $AC$, in Fig. 2, represent this speed. Then, if from $C$ we lay off $CD$ *up-stream* equal in length to the speed of the current, $AD$ will represent in length and direction the required velocity relative to the water: for $AC$ is the resultant of $AD$ and $DC$, which last represents the velocity of the current.

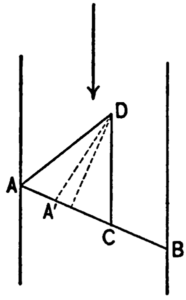

FIG. 2.

The geometrical subtraction effected in the process is equivalent to the addition to $AC$ of the vector $CD$, which is the *negative* of the vector to be subtracted.

33. Graphic solutions may also be given for problems concerning velocities in which the data do not consist of completely given vectors. For example, in Fig. 2, while $AB$ is still the direction of the required resultant, suppose that the rate of rowing were given instead of the resultant speed. Then, drawing the vector $CD$ from any point $C$ of $AB$, we find the point $D$. From this point as a centre, with a radius equal to the given speed of rowing, describe an arc cutting $AB$ in the point $A'$; then $A'D$ determines the proper direction of rowing. The construction shows that the least possible speed of rowing is represented by the perpendicular from $D$, and that in general there are two solutions giving different resultant speeds along $AB$.

## Resolution of Velocities.

34. A given velocity is readily resolved into two *components* having the directions of any two straight lines lying in the same plane with the given line of motion. To do this, it is only necessary to draw parallels to the given lines through the two extremities of the vector representing the given velocity. Thus, in Fig. 3, $OX$ and $OY$ being the given lines, the velocity $AC$ is by drawing the parallels $AE$ and $CE$ resolved into the component velocities $AE$ and $EC$.

When considering a number of velocities in one plane, we may thus, by adopting two intersecting straight lines as axes, replace each velocity by its two components in the directions of the axes. If we adopt a positive direction upon each axis, and take the component velocities along the axes themselves, it is apparent that the algebraic sum of the components along either axis of two given velocities is the like component of their resultant. Thus, in Fig. 3, $AC$ and $AB$ or its equal $CD$ being two given velocities, $A'C'$ and $C'D'$ are the components along $OX$, and their sum $A'D'$ is the like component of the resultant velocity $AD$. Again, along the axis $OY$ the components are $A''B''$ and $B''D''$, of which the latter is negative; accordingly their algebraic sum $A''D''$ is the like component of the resultant $AD$.

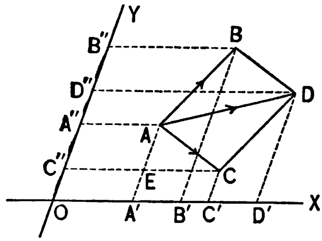


FIG. 3.

A velocity in a given plane is *determined* just as well by means of its two components along given axes as by means of its magnitude and direction, and the advantage of using this system consists in its simplification of the relation between given velocities and their resultant.

## Motion in a Plane Curve.

35. When a particle moves in a curve, the direction of its velocity at any instant is that of the tangent to the curve; and the vector representing the velocity is a portion of this tangent, measured in the direction of the motion, and equal in length to the speed or numerical measure of the velocity, which we shall denote by $v$. Thus, in Fig. 4, if a particle is moving in the curve $AB$, and $AC = v$ be measured off upon the tangent, it will be the vector representing the velocity. In this position, the vector $AC$ is *the space which the particle would describe in the next unit of time if during that interval its velocity remained the same both in amount and direction as it is at the point $A$.*

Now, since the particle moves in a curve, this vector is variable, because its direction varies, even if its magnitude remains constant. For example, when the particle has arrived at $B$ its velocity will be represented by a certain vector $BD$, which will generally differ in direction from $AC$, whatever be the relative magnitudes of the lines.

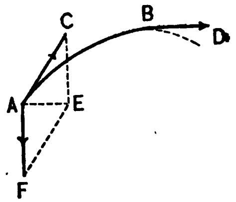


FIG. 4.

36. In order to compare the velocities at $A$ and $B$, we may draw a vector $AE$ from $A$ equal to the vector $BD$, that is, parallel to $BD$ as well as equal to it in length. The vector $CE$ will then represent the total change of velocity which the particle undergoes in passing from $A$ to $B$ (when direction as well as magnitude is taken into account), because it is the vector which must be geometrically added to $AC$ in order to produce $AE$. Completing the parallelogram, we may also take the vector $AF$ to represent the change of velocity; it is in fact vectorially equal to $AE + EF$, that is, to $BD - AC$.

## The Hodograph.

37. In order to compare the velocities at all points, in a case of plane curvilinear motion, the vectors representing the several velocities may all be laid off from a common point taken for convenience in a separate diagram. For example, suppose a particle to describe the ellipse $ABC$, in Fig. 5, with variable speed. From any point $O$ let $OA'$ be drawn parallel and equal to the vector representing the velocity at $A$. From the same point let $OP'$ be drawn equal and parallel to the velocity at any point $P$. By supposing the point $P$ to move continuously about the ellipse, the point $P'$ describes a curve, which will be a closed curve if, as supposed in the figure, the particle arrives at $A$ with the same velocity with which it started. This curve is called *the hodograph* of the given motion. The point $O$ is called *the pole*; and it must be remembered that, in order to represent a given motion of $P$, the hodograph must be taken in connection with a certain pole.

Moreover, for any given motion of $P$, the auxiliary point $P'$ will have *a certain corresponding law of motion.* For instance, the hodograph of a motion at uniform speed in any curve whatever would be a circle referred to its centre as pole; but the motion of $P'$ in this circle would depend on the curvature of the path of $P$.

38. It follows from the construction of the hodograph that the vector $A'P'$ represents the change in velocity experienced by the particle in moving from $A$ to $P$ in Fig. 5, just as $CE$ does in Fig. 4. So, in general, the change of velocity in any arc of the given motion is represented by the chord of the corresponding arc of the hodograph. Thus the displacement or change of *position* of $P'$ indicates the continuous change in the *velocity* of $P$.

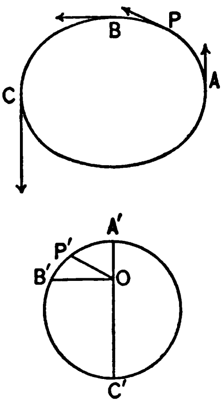

FIG. 5.

### Acceleration in Curvilinear Motion.

39. By an extension of its original meaning, the term *acceleration* is used to denote *the rate of change of velocity when direction as well as speed is considered.* Hence, when the hodograph is constructed, the acceleration of $P$ is the rate of displacement of $P'$; that is, *the velocity of the auxiliary point in the hodograph.*

Thus, acceleration in general is a vector quantity, that is, one having direction as well as magnitude; and whenever the hodograph is a curve, it is variable in direction at least. Its magnitude is the same as the speed of the auxiliary point $P'$.

40. As an example, let us consider the case of *uniform circular motion.* Let $C$, Fig. 6, be the centre of the circular path, and $a$ its radius. Denote by $V$ the constant value of the speed which is the length of the vectors $AA'$, $PP'$. The hodograph, constructed as in Art. 37, is therefore a circle whose radius is $V$. Moreover, since the vectors $AA'$, $PP'$ are perpendicular to the radii $CA$, $CP$, the angles $ACP$, $A'OP'$ are equal. It follows that

the point $P'$ moves uniformly in the hodograph, completing a revolution in the same time that $P$ does. The vector $\alpha$, constructed in the diagram to represent the velocity of $P'$, represents also (Art. 39) the acceleration of $P$. Since this vector is perpendicular to the radius $OP'$ of the hodograph, it is parallel to $PC$; hence the acceleration is constant in magnitude and is directed toward the centre of the circle in which the particle moves. Now the velocities are proportional to the radii, because the two circles are described in the same time, therefore $\alpha : V = V : a$; whence

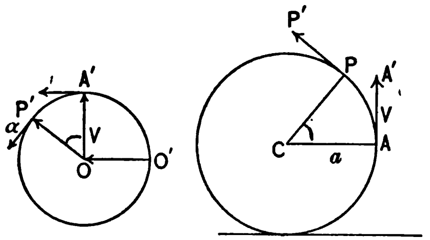

FIG. 6.

$$\alpha = \frac{V^2}{a}$$

gives the magnitude of the acceleration.

41. Consider next the motion of a point in a circle rolling upon a straight line. For example, suppose the circle in Fig. 6 to be rolling, like a carriage-wheel, with uniform speed upon a horizontal tangent. The wheel then has a motion of translation toward the left with the speed $V$, and any point of the rim, as $P$, has in addition a uniform circular motion relatively to the carriage. The velocity of $P$ at any point is now the resultant of this constant horizontal velocity and that which the point has by virtue of the rotation of the wheel. This is completely represented in the hodograph by removing the pole to the point $O'$, at the extremity of the horizontal radius; for the vector $O'P'$ is the resultant of $O'O$, representing the motion of translation of the carriage, and $OP'$ representing the relative velocity of $P$. It thus appears that the hodograph and the velocity of $P'$ in it are not affected by the constant velocity of translation, the only effect being to remove to a new position the pole of reference.

Thus the hodograph of uniform cycloidal motion is a circle referred to a point on its circumference as pole, and the acceleration in rolling motion is directed toward the centre and has the same magnitude as in uniform circular motion.

### Component Accelerations.

**42.** Referring the motion, as in Art. 34, to coordinate axes, $CD$, in Fig. 3, is the change taking place when the velocity $AC$ is changed to $AD$, whence it is easily seen that *the component along either axis of any change of velocity is the change in the like component of the given velocity.* The coordinates of the moving particle being $x$ and $y$, these component velocities are denoted by

$$\frac{dx}{dt} \quad \text{and} \quad \frac{dy}{dt}.$$

It follows that the *rates of change* of these component velocities, namely,

$$\frac{d^2x}{dt^2} \quad \text{and} \quad \frac{d^2y}{dt^2},$$

are the components along the axes of the acceleration. They are also called *the component accelerations*, and the actual acceleration, which is their resultant, is sometimes called in distinction *the total acceleration.*

**43.** In the analytical treatment of questions of motion rectangular coordinates are nearly always employed. Then, denoting by $s$ the length of the path as measured from some fixed point of it, and by $\phi$ its inclination to the axis of $x$, we have for the component velocities

$$\frac{dx}{dt} = \frac{ds}{dt}\cos\phi = v\cos\phi, \quad \frac{dy}{dt} = \frac{ds}{dt}\sin\phi = v\sin\phi, \quad . \quad (1)$$

where $v$ is the actual speed of the point; whence

$$v = \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}. \quad . \quad . \quad . \quad . \quad (2)$$

and

$$\tan\phi = \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

give the resultant velocity and its direction in terms of the component velocities.

In like manner, if $\psi$ denotes the inclination of the acceleration $\alpha$, the component accelerations are

$$\frac{d^2x}{dt^2} = \alpha \cos \psi, \qquad \frac{d^2y}{dt^2} = \alpha \sin \psi; \quad . \quad . \quad . \quad (4)$$

and

$$\alpha^2 = \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\tan \psi = \frac{d^2y}{dt^2} \Big/ \frac{d^2x}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

give the total acceleration and its inclination, in terms of the component accelerations.

44. As an illustration, we give the analytical treatment of the case of uniform circular motion which has been treated graphically in Art. 40. Taking the centre of the circle, $O$, in Fig. 7 as origin of rectangular axes, denote by $\theta$ the angle $POA$ made with the axis of $x$ by the radius $OP$ at the time $t$. Then, supposing $A$ to be the position corresponding to $\theta = 0$, $\theta = \omega t$, where $\omega$ is a constant because the motion is uniform. It is in fact the *angular velocity* of $P$. The coordinates of $P$ are

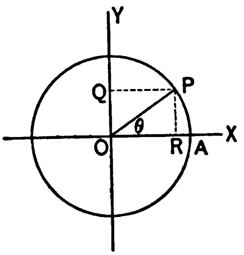

FIG. 7.

$$x = a \cos \theta = a \cos \omega t, \qquad y = a \sin \theta = a \sin \omega t, \ . \ (1)$$

whence, differentiating, the component velocities are

$$\frac{dx}{dt} = - a\omega \sin \omega t, \qquad \frac{dy}{dt} = a\omega \cos \omega t. \ . \ . \qquad (2)$$

Therefore, by equations (2) and (3), Art. 43, the linear velocity $v$ and its direction are given by

$$v^2 = a^2\omega^2, \qquad \tan \phi = - \cot \theta; \ . \ . \ . \ . \ (3)$$

whence, measuring $s$ from $A$, so that $v$ is positive when $\theta$ increases,

$$v = \frac{ds}{dt} = a\omega, \qquad \phi = \theta + 90^\circ.*$$

Again, differentiating equations (2), the component accelerations are

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t, \qquad \frac{d^2y}{dx^2} = -a\omega^2 \sin \omega t;$$

whence, in like manner, equations (5) and (6), Art. 43, give

$$\alpha = a\omega^2 \qquad \text{and} \qquad \psi = \theta + 180^\circ.$$

Since $v = a\omega$, this value of $\alpha$ agrees with the result of Art. 40, and the value of $\psi$ shows that the acceleration is directed toward the centre.

It is to be noticed that the resolved velocities of a point $P$ along two rectangular axes are the actual velocities of the points $R$ and $Q$, Fig. 7, which are called *the projections of $P$* upon the axes. In the present case, the motion of each of these points is the harmonic motion discussed in Art. 11. Harmonic motion is in fact often defined as the motion of the projection of a point in uniform circular motion.

## Application of the Second Law of Motion to Forces in Different Directions.

45. The Second Law of Motion, namely, that: *Change of motion is proportional to the moving force acting, and takes place in the straight line in which the force acts*, implies that, when several forces are acting upon the same body, each one produces a proportional change of motion in its own direction.

* The angle $\phi$ as determined by equation (3) is $\phi = \theta \pm 90$, but the ambiguity is removed by equations (2).

We have seen, in the first section of this chapter, how this change of motion in the case of a single force is to be estimated, and the additional application of the law now to be made may be expressed thus: *The motions which forces in different directions would produce if acting singly on a body coexist in their joint action.*

46. There are two modes in which we may regard the joint action of forces in the directions of two intersecting lines. In the first place, suppose a particle at rest at $A$, Fig. 8, to be acted upon by a force whose total action or impulse (see Art. 22), communicated suddenly, would give it the velocity $AB$ (that is, cause it to move from $A$ to $B$ in the unit of time) and at the same time by a force whose total action would give it the velocity $AC$; then, by the second law, it will by the joint action receive a motion which will cause it to move to $D$, the opposite vertex of the parallelogram $ABDC$, in the unit of time; therefore the joint action will give it the velocity $AD$ in the direction of the diagonal. The impulse which acting alone would produce this joint effect is called the *resultant* impulse. By Art. 22, impulses are measured by the momenta they produce; therefore the given and resultant impulses, which have a common mass factor, are proportional to, and in the direction of, the lines $AB$, $AC$ and $AD$. It follows that, if two impulses are represented by proportional straight lines in the proper directions, their resultant will be represented by the diagonal of the parallelogram of which they are the sides.*

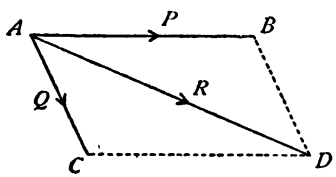

FIG. 8.

*This is equivalent to Newton's proof of the "parallelogram of forces" (*Corollaria I* and *II* of the *Axiomata sive Leges Motus*); the forces being, as mentioned in the foot-note to Art. 22, what we now call impulses. The result applied also to the "continuous forces" (*vires acceleratrices*) of the *Principia*, because these were measured by the actions produced in a given time.

47. In the second mode of regarding the joint action of two given forces, we may suppose the particle of mass $m$ at $A$, Fig. 8, to be already moving in any manner whatever, while $AB$ and $AC$, drawn in the directions of the two forces, are vectors representing the *accelerations* which the two forces each acting singly would produce in the mass $m$. Then, by the second Law of Motion, the joint action of the forces is to produce in the particle the acceleration represented in amount and direction by the diagonal $AD$ of the parallelogram. But this acceleration would be produced by a single force of the proper amount in the direction $AD$. The measures of the given forces and of the single force, which is called *the resultant*, are, by Art. 17, $mAB$, $mAC$ and $mAD$, respectively. Hence, if two forces acting upon a particle are represented by vectors having their directions and having lengths proportional to their magnitudes, the single equivalent force, or *resultant*, will be represented in direction and magnitude by the diagonal of the parallelogram, or vectorial sum of the two vectors.

The construction, when thus applied to forces, is known as the *parallelogram of forces*. It is to be noticed that, since the value of $m$ is arbitrary, the scale in which forces are represented by lengths is purely arbitrary

## Momentum as a Vector Quantity.

48. Momentum being the product of mass and velocity has, like the latter, a definite direction : in other words, it is a vector quantity. The parallelogram of forces shows that, when forces act simultaneously upon a body, each may be regarded as producing a momentum in its proper direction, and that these momenta coexist subject to the law of composition of vectors. This is readily seen to extend to any number of forces. The momenta produced may exactly neutralize one another, and in that case no change of motion will take place, so that the body will either be at rest or moving uniformly in a straight line. In such a case, the forces are said to be *in equilibrium*.

## EXAMPLES. II.

1. A body undergoes three displacements, of 1, 2 and 3 units respectively, in the directions of a point describing the three sides of an equilateral triangle. What is the resulting displacement?

$\sqrt{3}$, in a direction perpendicular to the second side.

2. A ship is carried by the wind 3 miles due north, by the current 2 miles due west, and by her screw 6 miles E. 30° S. What is her actual displacement? and if these displacements take place uniformly in half an hour, what is the velocity relative to the water? $(3\sqrt{3}-2)$ miles E.: $6\sqrt{3}\,^{m}/_{h}$.

3. The hood of a market van is $3\frac{1}{2}$ feet above the floor: in driving through a shower the floor is wet to a distance of 11 inches behind the front edge of the hood. Assuming the rain-drops to fall vertically with a uniform velocity of $28\,^{f}/_{s}$, what is the rate of driving? 5 miles per hour.

4. A man jumps with a velocity of 8 feet per second from a car running ten miles an hour, in a direction making an angle of 60° with the direction of the car's motion. With what velocity does he strike the ground? $\frac{4}{3}\sqrt{223}=19.91\,^{f}/_{s}$.

5. Two ships, A and B, are approaching with uniform speeds the intersection of the straight lines in which they move. If the bearing of B from A is unchanging, show that the velocity of B relative to A is in the opposite direction, and that the ships will meet. If the speeds remain fixed and the courses vary, what is the locus of the point of meeting?

6. A ship is steaming in a direction due north across a current running due west. At the end of an hour and a half it is found that the ship has made 24 miles in a direction 30° west of north. Find the velocity of the current, and the rate at which the ship is steaming. $8\,^{m}/_{h}$; $8\sqrt{3}\,^{m}/_{h}$.

7. A street car is moving with the speed of 9 miles per hour. At what inclination to the line of motion must a package be projected from it, with a velocity of 24 feet per second, in

order that the resultant motion may be at right angles to the track? $\cos^{-1}(-\frac{11}{30}) = 123^\circ\ 22'$.

8. Assuming that the earth moves in a circular orbit about the sun, and that light travels from the sun to the earth in $8^m\ 20^s$, find the apparent displacement of the sun due to the earth's motion. $20''.55$.

9. A point is moving eastward with a velocity of $20^f/_s$, and one hour afterwards it is found to be moving northeast with the same speed. Find the change of velocity, and the measure of the acceleration, if the latter is assumed to be uniform.

$20\sqrt{(2-\sqrt{2})}\,^f/_s$ N. N. W.; $\frac{1}{180}\sqrt{(2-\sqrt{2})}$.

10. Assuming the labor of rowing for a given time to be proportional to the square of the speed, and denoting the angle between $AB$ (see Fig. 2) and the direction of the stream by $\phi$, show that the labor of rowing from $A$ to $B$ is a minimum when the direction of rowing makes with $AB$ the angle $90^\circ - \frac{1}{2}\phi$.

11. A carriage is travelling at the rate of six miles an hour. What is the velocity in feet per second of a point midway between the centre and rim of the wheel: ($\alpha$) at its highest; and ($\beta$) at its lowest point? ($\alpha$) 13.2; ($\beta$) 4.4.

12. A point is describing a circle, of radius 7 yards, in 11 seconds with uniform speed. Find the change in its velocity after describing one sixth of a revolution from a given initial point.

About $12^f/_s$ at an angle of $120^\circ$ with the initial motion.

13. A train is travelling at the rate of 45 miles per hour, and rain is driven by the wind, which is in the same direction as the motion of the train, so that it falls with a velocity of 33 feet per second at an angle of $30^\circ$ with the vertical. Show that the apparent direction of the rain to a person in the train is at right angles to its true direction.

14. A train moving at the rate of 30 miles per hour is struck by a stone moving horizontally and at right angles to the track with a velocity of 33 feet per second. Find the magnitude of the velocity with which the stone strikes the train, and the angle it makes with the motion of the train.

$55^f/_s$; $\tan^{-1}(-\frac{3}{4})$.

15. A ship is sailing due east, and it is known that the wind is blowing from the northwest; the apparent direction of the wind as shown by the pennant is from N. N. E. Show that the velocity of the ship is equal to that of the wind.

16. A person walking eastward at the rate of $3^m/_h$ finds that the wind seems to blow directly from the north, and on doubling his speed it seems to blow from the northeast. Find the velocity and direction of the wind. $3\sqrt{2}^m/_h$ from N. W.

17. What is the amount and direction of the momentum received by a body of mass $m$ moving uniformly with velocity $v$ in a circle: ($\alpha$) in a half-revolution; ($\beta$) in a quarter-revolution?

($\alpha$) $2mv$ opposite the original direotion;
($\beta$) $mv\sqrt{2}$ at an angle of $135°$.

18. If the speed of a carriage be represented by the radius of the wheel, show that the velocity of a point on the rim at any instant is represented in length by the chord joining it with the point in contact with the ground, and is perpendicular to this chord.

19. Derive the acceleration in the case of the uniformly rolling wheel (Art. 41) from the equations of the cycloid.

20. Draw the hodograph for a point of the wheel midway between the centre and the rim, and thence show that the greatest inclination of the velocity of this point to the horizontal is $30°$.

21. Show that, if one of two component velocities of a point in fixed directions is constant, the hodograph of the motion is a straight line.

22. A bicycle is "geared to $2b$ inches," and the length of the crank is $a$ inches. Determine the arc $\phi$ in which back-pedalling is effective on a down grade whose inclination to the horizon is $\alpha$; also the value of $\alpha$ for which $\phi$ vanishes.

$$\cos \tfrac{1}{2}\phi = \frac{b \sin \alpha}{a}; \quad \alpha_1 = \sin^{-1}\frac{a}{b}.$$

23. If in Fig. 5 we assume the motion in the ellipse to be such that the hodograph is a circle (the pole $O$ *not* at the centre), show that the accelerations at the extremities of a diameter $PQ$

of the ellipse will make supplementary angles with the velocities at these points.

24. If two bodies connected by a string revolve uniformly about one another there is a point of the string which is at rest; show, by the third law of motion, that the accelerations are inversely proportional to the masses, and thence that the point at rest divides the string in the inverse ratio of the masses.

# CHAPTER II.

## FORCES ACTING AT A SINGLE POINT.

### III.

### Statics.

49. The part of the Science of Mechanics upon which we now enter is concerned only with the tendencies to action of given forces at any instant, and not with the motions produced. It is called *Statics* because the bodies on which the forces act are assumed to be at rest.

We have seen in Art. 47 that, when two forces are acting upon a particle, there exists a single force, called their resultant, to which they are statically equivalent; and that, representing the given forces by vectors, that is to say, by straight lines drawn in their directions and proportional in length to their magnitudes, the resultant is in like manner represented by the diagonal of the parallelogram of which these lines are adjacent sides.

By the principle of transmission of force (Art. 27), the point of application of a force acting upon a rigid body may, so far as the immediate action of the force is concerned, be transferred to any point of the line of action. Thus, in statics, we have only to consider the line of action and the magnitude of a force. Hence, when two forces act upon a rigid body, *if their lines of action intersect*, they may be regarded as acting at the point of intersection, and *they have a resultant* which acts at this point, and is found in direction and magnitude by the parallelogram of forces.

But, if the lines of action do not intersect, it does not follow,

and in fact is not generally true, that there is a single force or resultant equivalent to the two forces.

In the present chapter, the forces will be regarded as acting upon a single particle, or at a single point of a rigid body through which all the lines of action pass.

## The Resultant of Two Forces.

**50.** Let $AB$ and $AC$, Fig. 9, represent two given forces, $P_1$ and $P_2$, acting at the point $A$, and let $\phi$ denote the angle $BAC$ between their directions. The direction of the resultant divides the angle $\phi$ into two parts, $\theta_1$ and $\theta_2$. In the triangle $ABD$, the angle $ABD$ is the supplement of $\phi$, and $BD = AC$; hence

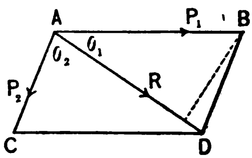


FIG. 9.

$$R^2 = P_1^2 + P_2^2 + 2P_1P_2 \cos \phi, \quad . \quad . \quad . \quad (1)$$

which gives the magnitude of $R$ in terms of the given quantities.

The angle $ADB = \theta_2$; hence, from the same triangle,

$$P_1 : P_2 : R = \sin \theta_2 : \sin \theta_1 : \sin \phi; \quad . \quad . \quad . \quad (2)$$

that is, the angle $\phi$ is cut by the resultant into parts whose sines are inversely proportional to the adjacent forces.

It is obvious also, on drawing a perpendicular from $B$ upon $AD$, that

$$R = P_1 \cos \theta_1 + P_2 \cos \theta_2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In particular, if the given forces are equal, a case of frequent occurrence, the angle $\phi$ is bisected; and putting $P_1 = P_2 = P$, we have for the resultant of two equal forces making the angle $\phi$

$$R = 2P \cos \tfrac{1}{2}\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

## Statical Verification of the Parallelogram of Forces.

**51.** In making the statical comparison of forces, we have frequent occasion to make use of the transmission of force (Art. 27) without change of magnitude by means of a flexible string, in

which case the magnitude of the force is the *tension* of the string. It is necessary also to have a method of changing the direction without altering the magnitude of a force. This is accomplished by supposing the string to pass round a smooth peg or pulley. Thus, if the string $AB$, Fig. 10, passes round the fixed smooth peg at $C$, and is subjected at the end $B$ to the pull of a suspended weight $W$ and at $A$ to a pull equal to $W$ (so that the tension of each part of the string is $W$), it will not slip in either direction, because there is no reason why it should move in one direction rather than the other. Conversely, if the string does not slip, and *the peg is perfectly smooth*, the tensions of the two parts must be equal; for the action of the smooth surface is, at every point of the arc of contact, perpendicular to the direction of the string, and therefore cannot disturb the equality of the two opposite forces which balance each other at the point.

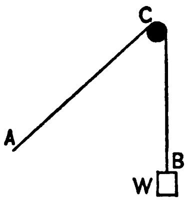

FIG. 10.

52. Using this method of changing the direction of a force, we can make an experimental verification of the parallelogram of forces as follows: Let three weights $P$, $Q$ and $R$, of which no one exceeds the sum of the other two, be attached to three strings knotted together at $C$, and let the strings attached to $P$ and $Q$ pass over smooth pegs $A$ and $B$ fixed in a vertical wall, as in Fig. 11. Let the weights now be allowed to adjust themselves so as to be at rest. In the position of rest, the forces $P$ and $Q$ act, as in Art. 51, obliquely at the knot $C$ in the directions $CA$ and $CB$, and their resultant must be a force equal to $R$ acting vertically upward, because it sustains the weight $R$ acting vertically downward. Now, if a point $D$ be taken upon $CR$ pro-

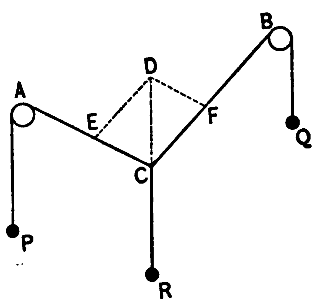

FIG. 11.

duced, and the parallelogram $DECF$ be completed, it will be found, on measurement, that the lines $CE$, $CF$ and $CD$ have the same ratios as the weights $P$, $Q$ and $R$, which agrees with the principle of the parallelogram of forces.

## Three Forces in Equilibrium.

53. Three forces which, as in Fig. 11, act upon a point which remains at rest are said to be *in equilibrium*. It is obvious that the resultant of any two of the three forces is a force equal to the third, but opposite to it in direction. Such forces therefore have lines of action *lying in one plane*, and in magnitude they are proportional to the sides and diagonal of a parallelogram drawn as in Fig. 11; or what is the same thing, to the sides $CE$, $ED$, $DC$ of *a triangle*, such as $CED$ in Fig. 11, *whose sides are in or parallel to the lines of action.*

Such a triangle is called a *triangle of forces* for the equilibrium of the particle on which the forces act. Thus, in Fig. 11, either $DCE$ or $DCF$ may be taken as the triangle of forces for the equilibrium of $C$.

54. The *directions* of the forces are those in which a point must move in describing the complete perimeter of the triangle in one direction; for example, in the direction $DCE$ in the first of the triangles mentioned above, or $DCF$ in the other. The angles between the directions of the forces are the supplements of the angles of the triangles, and therefore have the same sines. It follows that

$$P : Q : R = \sin BCR : \sin RCA : \sin ACB;$$

that is to say, three forces in equilibrium are proportional each to the sine of the angle between the other two.

When a rigid body is in equilibrium under the action of three forces, they may have different points of application, but *their lines of action must lie in one plane and must meet in one point.* Whenever a triangle of forces is drawn, it must be remembered that the forces do not act *along* the three sides of the triangle, but in lines parallel to them which, if there are no other forces acting, meet in a single point.

## Resolution of Forces.

55. If, in any plane containing the line which represents a given force, lines be drawn through its extremities in any given directions, a triangle will be formed the sides of which represent in magnitude and direction *two forces of which the given force is the resultant.* These forces are called *components* of the given force. Thus, in Fig. 12, if $AB$ represents the given force, and $AC$ and $CB$ drawn in the given directions intersect at $C$, the lengths $AC$ and $CB$ represent in magnitude the two components of $AB$. The given force is then said to be *resolved* into a pair of components in given directions. But, if $AC$ is taken as the line of action of one component, the component represented in direction and magnitude by $CB$ must act in the parallel line $AD$.

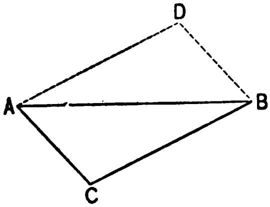


FIG. 12.

If the given force is regarded as acting upon a rigid body, it may, of course, be resolved into similar components both acting at $B$, and in the lines $CB$ and $DB$ respectively.

## Effective or Resolved Part of a Force in a Given Direction.

56. The component of a force in a given direction is not determined unless the direction of the other component is given. Thus, in Fig. 12, $AC$ may be drawn in the given direction; but its length (representing the magnitude of the component of $AB$ in this direction) is not determined unless we know the direction in which to draw $BC$. Now, in many cases, what we require is the effectiveness of the force to produce motion in the direction in question when no other motion can take place. Suppose, for example, that a particle at $A$, Fig. 13, is confined in a smooth tube $AC$, or constrained in

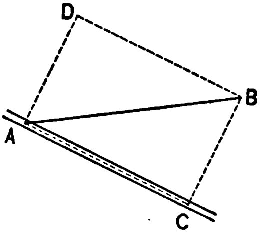


FIG. 13.

some other manner, so that it can move only along the line $AC$. If now the particle is acted upon by the force $AB$ it will move along the tube in the direction $AC$. The force which prevents any other motion is now the resistance of the tube, which, since the tube is supposed to be perfectly smooth, acts at right angles with it. Taking this as the direction of the other component, as represented in the figure, we have $AC$, one of *two rectangular components*, as the measure of *the action of the force* in the given direction.

57. This rectangular resolution of a force is of such importance that the rectangular component in a given direction is generally referred to simply as *the resolved part* of the force in the given direction; and this is always to be understood by the term "resolved part" or "component," when no other direction is mentioned.

The length $AC$, in Fig. 13, is *the projection* of $AB$ upon the line $AC$, and, denoting the angle $BAC$ by $\theta$, its value is $AB \cos \theta$; hence the resolved part of a force $P$ in the direction of a line making the angle $\theta$ with its line of action is

$$P \cos \theta.$$

Equation (3), Art. 50, expresses that the sum of the resolved parts of two given forces in the direction of their resultant is the resultant itself. It is readily seen also from Fig. 9 that the resolved parts of the given forces in a direction at right angles to their resultant are equal and opposite forces. These components counterbalance one another, so that there is no force tending to move the body to either side of the diagonal $AD$.

### The Resultant of Three or more Forces.

58. Let $P_1$, $P_2$ and $P_3$, Fig. 14, represent three forces acting on a particle at $O$. If from any point $A$ we lay off $AB$ equal and parallel to $P_1$, and then from $B$ lay off $BC$ equal and parallel to $P_2$, we shall have determined (without completing the parallelogram) the point $C$, such that $AC$ represents in direction and

magnitude the resultant of $P_1$ and $P_2$. Denoting this resultant by $Q$, we shall in like manner, by laying off from $C$ $CD$ equal and parallel to $P_3$, arrive at the point $D$, such that $AD$ represents in direction and magnitude the resultant of $Q$ and $P_3$. Now the joint action of the three forces is the same as that of $Q$ and $P_3$ acting at $O$, and therefore is equivalent to the action of a single

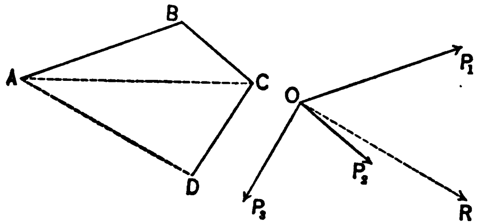

FIG. 14.

force $R$ acting at $O$ and represented in magnitude and direction by $AD$. This force is called *the resultant* of the three forces $P_1$, $P_2$ and $P_3$.

59. The forces are here represented by *vectors*, and the process is an extension of the geometrical or vectorial addition mentioned in Art. 29. Since the order in which any two of the vectors are taken is immaterial, we arrive at the same final point $D$, whatever be the order of geometrically summing the three vectors.

The three forces may or may not lie in one plane. When they do not, $P_1$, $P_2$ and $P_3$ at $O$ may be regarded as three edges of a parallelopiped of which the diagonal from $O$ represents the resultant. The different orders in which the vectors can be added then correspond to the different paths by which a point might move from $O$ to the opposite vertex passing over three edges of the parallelopiped.

60. The process of Art. 58 is evidently applicable to any number of forces. When the final point arrived at in the geometrical addition coincides with the initial point, the resultant is zero, and the forces are said to be *in equilibrium*. For example, we shall have such a system of forces if, in Fig. 14, in addition to $P_1$, $P_2$ and $P_3$, there were acting at $O$ a fourth force equal and

opposite to the resultant $R$, which would be vectorially represented by $AD$. The closed perimeter such as $ABCDA$ formed in this case is known as *the polygon of forces*, and the theorem is that: *If any number of forces acting at a point are represented in direction and magnitude by the sides of a closed polygon, each taken in the direction of the motion of a point describing the complete perimeter, the forces are in equilibrium.*

If the lines of action of the forces are not all in one plane, the theorem still holds, the polygon of forces being, in that case, not a plane figure, but what is called a *skew polygon.*

## The Resolved Part of the Resultant.

**61.** If, through the extremities of a vector $AB$, planes perpendicular to a given line be passed, the length which they intercept on this line is called the *projection* of $AB$ upon the given line. With this definition, the projection of $AB$ upon any two parallel lines is the same; for the same projecting planes are used, and the projection is the perpendicular distance between these planes. The line $MN$, Fig. 15, upon which the projection $A'B'$ is

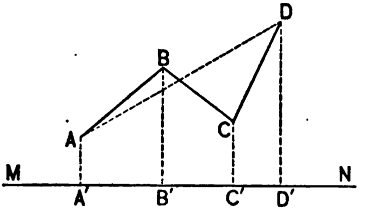

FIG. 15.

made, may not be in a plane with $AB$; but the lines $AA'$, $BB'$ will in all cases be perpendicular to $MN$. If now we define the inclination of two lines which do not intersect as the same as the angle $\theta$ between intersecting lines parallel to them, we shall have, as in Art. 57, for the length of the projection of $AB$,

$$A'B' = AB \cos \theta.$$

**62.** Now if we take the broken line $ABCD$, formed in the vectorial addition of the forces $P_1$, $P_2$ and $P_3$, Fig. 15, and pass planes perpendicular to $MN$ through $A$, $B$, $C$ and $D$, we see that $A'D'$, the projection of the resultant $AD$, is the algebraic sum of the projections of the vectors $AB$, $BC$ and $CD$. Denoting by $\theta_1$, $\theta_2$, etc., the angles between the direction taken as positive along

*MN* and that of the forces respectively, $P_1 \cos \theta_1$, etc., (Art. 57,) express the resolved parts along *MN* of the given forces (which are represented by the projections) with their proper signs (the projection being negative when $\theta$ is obtuse). Hence, if $\phi$ is the inclination of the resultant $R$,

$$R \cos \phi = P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3.$$

The result of course extends to any number of forces; that is to say, *the resolved part of the resultant of a number of forces in any direction is the algebraic sum of the like resolved parts of the given forces.*

### Reference of Forces in a Plane to Coordinate Axes.

**63.** In the systematic treatment of forces in a plane, coordinate axes are assumed, and the components of a force $P$ along the axes of $x$ and $y$ respectively are denoted by $X$ and $Y$.

The demonstration applied in Art. 34 to velocities shows that, for any quantities represented by vectors and which combine by the vector law, the component of the resultant in the direction of either axis is the sum of the like components of the given quantities. Thus, in Fig. 3, $AB$ and $AC$ may be taken to represent the forces $P_1$ and $P_2$, and $AD$, their resultant $R$. Hence the components of $R$ are

$$X_1 + X_2 \quad \text{and} \quad Y_1 + Y_2.$$

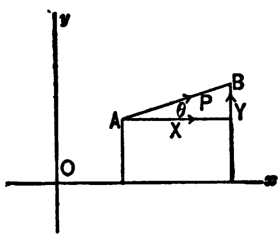


FIG. 16.

In like manner, for any number of forces, using the sign of summation $\Sigma$, the components of the resultant are

$$\Sigma X \quad \text{and} \quad \Sigma Y.$$

**64.** When, as is usually most convenient, rectangular axes are used, the components are "resolved parts" (Art. 57). Denoting by $\theta$ the inclination of the force $P$, Fig. 16, to the positive direction of the axis of $x$, the inclination to the positive direction of the axis of $y$ is the complement of $\theta$ and we have

$$\left.\begin{aligned} X &= P\cos\theta, \\ Y &= P\sin\theta. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

From these we derive for the determination of $P$ and $\theta$, when $X$ and $Y$ are given,

$$P^2 = X^2 + Y^2, \qquad \tan\theta = \frac{Y}{X}. \quad . \; . \; . \quad (2)$$

It follows that, if $\phi$ denotes the inclination of the resultant $R$ of any number of forces,

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2, \qquad \tan\phi = \frac{\Sigma Y}{\Sigma X}.$$

## Rectangular Components in Space.

**65.** When the forces under consideration do not all lie in one plane, a system of coordinate axes in space may be assumed, as in Fig. 17. Let the given force $P$ be represented by a line $OA$ drawn from the origin. Draw $AB$ parallel to the axis of $y$ to meet the plane of $xz$, and join $OB$. Then $OB$ and $BA$ represent in magnitude and direction a pair of components of the given force. It will be noticed that in general the component in a given plane is not determined either in magnitude or direction unless the direction of the other component is known. But, when this other component is perpendicular to the given plane, as $AB$ in Fig. 17, where the axes are supposed rectangular, $OB$ is a definite line known as *the projection of OA upon the plane*, and the component it represents is called *the resolved part of P in the plane.*

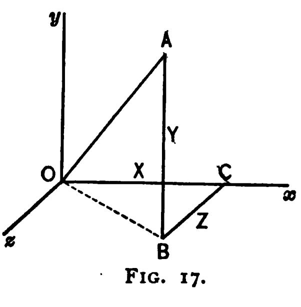

FIG. 17.

The magnitude of this resolved part is $P$ cos $AOB$, where $AOB$ is the inclination of the force to the plane.

**66.** Again, resolving this component in the direction of rectangular axes of $x$ and $z$, by drawing $BC$ parallel to the axis of $z$, we have $P$ resolved into three components represented in magnitude and direction by $OC$, $CB$ and $BA$, each of which is the resolved part in the direction of one of the axes.

Denoting by $\alpha$, $\beta$ and $\gamma$ the angles $AOx$, $AOy$ and $AOz$, or *direction angles* of $OA$, we have for the three components of $P$

$$X = P\cos\alpha, \quad Y = P\cos\beta, \quad Z = P\cos\gamma. \quad . \quad (1)$$

We also have $OA^2 = OB^2 + AB^2 = OC^2 + CB^2 + AB^2$; that is,

$$P^2 = X^2 + Y^2 + Z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The factors $\cos\alpha$, $\cos\beta$, $\cos\gamma$ in equations (1) are called *the direction cosines* of $P$ because they determine its direction. They are not, however, three independent quantities, but are equivalent to only two, for they are connected by the equation

$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma \quad . \quad . \quad . \quad . \quad (3)$$

derived from equations (1) and (2).

## Another Method of Constructing the Resultant.

**67.** The following theorem is sometimes useful in constructing the resultant of two forces: Given the triangle $ABC$, Fig. 18, and any two numbers, $m$ and $n$, the resultant of two forces represented in direction and magnitude by $mAB$ and $nAC$, will be represented in direction and magnitude by $(m + n)AD$, where $D$ divides $BC$ in the ratio $n : m$.

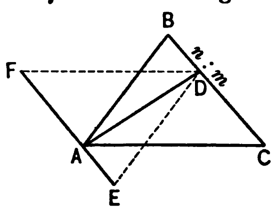


FIG. 18.

On a line parallel to $BC$, through $A$, lay off $AE$, $AF$, equal to and in the direction of $BD$, $CD$ respectively; then, by hypothesis, $mAE = nAF$. $AEDB$ and $AFDC$ are parallelograms; hence the resultant of $mAB$ and $mAE$ is $mAD$, and that of $nAC$ and $nAF$ is $nAD$. Thus the resultant of the four forces $mAB$,

$mAE$, $nAC$, $nAF$ is $(m+n)\,AD$; and, since the forces $mAE$ and $nAF$ are equal and opposite, they neutralize each other, so that $(m+n)AD$ is the resultant of $mAB$ and $nAC$, which was to be proved.

68. The theorem proved above leads to another method of graphically determining the resultant of several forces, as follows: Let $n$ forces acting at $O$ be represented by $OA_1$, $OA_2$, . . . $OA_n$, Fig. 19. Bisect $A_1A_2$ in $B$; then, by the theorem, $2OB$ is the resultant of $OA_1$ and $OA_2$. Join $B$ with $A_3$, and cut off $BC = \frac{1}{3}BA_3$; then, by the same theorem, $3OC$ is the resultant of $2OB$ and $OA_3$; that is, of $OA_1$, $OA_2$ and $OA_3$. In like manner, if we lay off $CD = \frac{1}{4}CA_4$, $4OD$ is the resultant of the first four forces, and so on. We finally reach a point $L$, such that $nOL$ is the resultant of the $n$ forces.

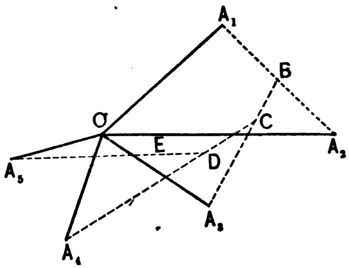


FIG. 19.

69. In this construction it is not necessary to suppose that the forces all lie in one plane. Whether the $n$ points $A_1$, $A_2$, . . . $A_n$ do or do not lie in one plane, the final point $L$ is called their *centre of position*. This point has the property that its distance from any plane is the *average distance* of the $n$ points from that plane. To prove this, denote the perpendiculars from the given points to the selected plane by $p_1, p_2, \ldots p_n$, and those from $B, C, \ldots L$ by $p_b, p_c, \ldots p_l$; then it is readily seen that the construction gives

$$p_b = p_1 + \tfrac{1}{2}(p_2 - p_1),$$
$$p_c = p_b + \tfrac{1}{3}(p_3 - p_b),$$
$$p_d = p_c + \tfrac{1}{4}(p_4 - p_c).$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

These equations, multiplied by 2, 3, 4, . . . $n$, give

$$2p_b = p_1 + p_2,$$
$$3p_c = 2p_b + p_3 = p_1 + p_2 + p_3,$$

$$4p_d = 3p_c + p_4 = p_1 + p_2 + p_3 + p_4,$$
$$\cdots\cdots\cdots\cdots$$
$$np_l = p_1 + p_2 + \ldots + p_n = \Sigma p;$$

hence $p_l = \dfrac{\Sigma p}{n}$, which is the average or *arithmetical mean* of the perpendiculars.

The theorem proved in the preceding article may now be stated thus: *If $n$ forces, $OA_1$, $OA_2$, . . . $OA_n$, act at $O$, their resultant is $n$ times the force represented by the line drawn from $O$ to the centre of position of $A_1$, $A_2$, . . . $A_n$.*

If the forces be in equilibrium, $O$ will itself be the centre of position of $A_1$, $A_2$, . . . $A_n$.

As an illustration, let the resultant be required of eight forces represented by lines joining a given point $O$ to the eight vertices of a parallelopiped. The centre of position of the eight vertices is obviously the centre of the figure; hence the resultant is represented by eight times the line joining $O$ to the centre of the parallelopiped.

## EXAMPLES. III.

1. Find the resultant of forces of 3 and 4 pounds, respectively, acting at right angles. 5 pounds.

2. At what angle must two equal forces act in order that the resultant shall equal either force? 120°.

3. Show that, if the resultant of two forces is equal to one of them, the forces act at an obtuse angle; also that, if the resultant is at right angles to one of the forces, it is less than the other.

4. Two forces when acting in opposite directions have a resultant of 7 pounds, and when acting at right angles they have a resultant of 13 pounds. What are the forces?

12 and 5 pounds.

5. Forces $P$ and $2P$ have a resultant at right angles to one of them. At what angle do they act? 120°.

6. If two of three forces in equilibrium are equal to $P$ and the angle between them is $\theta$, what is the other force?

$2P \cos \frac{1}{2}\theta$.

7. Forces of 5 and 3 pounds have a resultant of 7 pounds. At what angle do they act? $60^\circ$.

8. Forces of 3, 4, 5 and 6 pounds, respectively, act along the straight lines drawn from the centre of a square to the angular points taken in order. Find their resultant. $2\sqrt{2}$ pounds.

9. Three forces $P$, $2P$, $3P$ act at angles of $120^\circ$ to each other. Determine the resultant. $P\sqrt{3}$ at right angles to $2P$.

10. Lines parallel to the sides of a parallelogram intersect at a point $O$ within it. Show that the resultant of four forces at $O$ represented by the segments of these lines acts through the centre of the parallelogram.

11. Show that the resultant of three forces acting at the vertex $A$ of a parallelopiped and represented by the diagonals of the three faces meeting at $A$ is represented by twice the diagonal of the parallelopiped drawn from $A$.

12. $ABCDE$ is a regular hexagon; at $A$ forces act represented in magnitude and direction by $AB$, $2AC$, $3AD$, $4AE$, and $5AF$. Show that the length of the line representing their resultant is $AB\sqrt{351}$.

13. The chords $AOB$ and $COD$ of a circle intersect at right angles at $O$. Show that the resultant of forces represented by $OA$, $OB$, $OC$, $OD$ is represented by twice the line joining $O$ to the centre of the circle.

14. If $P$ is the *orthocentre* (point of intersection of the perpendiculars) of the triangle $ABC$, show that the resultant of forces acting at a point and represented in magnitude and direction by $AP$, $PB$ and $PC$ is represented by the diameter from $A$ of the circumscribing circle.

15. If $O$ is the centre of the circumscribed circle of the triangle $ABC$, and $P$ its orthocentre, show that the resultant of forces represented by $OA$, $OB$ and $OC$ is represented by $OP$.

16. If $P$ is the total pressure produced by the wind normal to the sails, supposed flat and making the angle $\theta$ with the keel, what is the effective force driving the ship ahead? $P \sin\theta$.

17. A weight $W$ is sustained by a tripod of equal legs, so

placed that the distance between each pair of feet is equal to a leg. Find the compression of each leg. $\dfrac{W}{\sqrt{6}}$.

18. $A$ and $B$, standing on opposite sides of a weight of 100 pounds, pull upon ropes attached to it and making angles of $45°$ and $60°$, respectively, with the horizontal. Find the ratio of their pulls if the resultant is vertical; also the value of $B$'s if the weight is just raised?

$1 : \sqrt{2}$; $100(\sqrt{3} - 1) = 73.2$ pounds.

19. If, in example 18, $B$'s rope be shifted to make an angle of $30°$ with the horizontal, show that his pull must be the same as before, but $A$'s must be multiplied by $\sqrt{3}$.

20. A force acting at $A$ is represented by the line $AB$. Show that the resolved part or action of the force in any direction $AC$ is represented by the chord from $A$ of a sphere whose diameter is $AB$; also, that the action in any plane through $A$ is represented by the diameter from $A$ of the small circle in which the plane cuts this sphere.

21. Show that if four forces in given directions which are *not* in one plane keep a particle in equilibrium, the ratios of the forces are determined; but if four forces in one plane, or more than four in general, are in equilibrium, their ratios are not determined by their given directions.

22. Show that, by resolving the system of forces $X$, $Y$, $Z$ in Fig. 17 along a line whose direction angles are $\lambda$, $\mu$, $\nu$, we obtain

$$\cos\psi = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu,$$

which is the expression for the cosine of the angle between two lines in terms of their direction cosines.

23. If the sides of the triangle $ABC$ be produced, namely, $BC$ to $D$, $CA$ to $E$, $AB$ to $F$, so that the parts produced are proportional to the sides, show that forces acting at a point and represented vectorially by $AD$, $BE$ and $CF$ are in equilibrium; also, if $O$ be any point in the plane, the forces $OD$, $OE$ and $OF$ have a resultant independent of the ratio $BD : BC$.

IV.

## Conditions of Equilibrium for a Particle.

**70.** As stated in Art. 25, when a body acted on by a single *active* force is prevented from moving by the resistance of a fixed body with which it is in contact, this *resistance* is regarded as a force equal and opposite to the active force, and thus, with it, producing equilibrium. So also, when a body acted upon by several known forces is kept at rest by bodies with which it is in contact, the resistances or *reactions* of these bodies are regarded as forces; and these, together with the known forces, constitute a system of forces in equilibrium.

**71.** In the case of a single particle, or of a body which may be regarded as such, the forces all act at a single point, and it is important to represent in the diagram *all* the forces, including the resistances. This is usually done by drawing lines *from* the point in the directions of the forces. For example, let it be required to find the force $P$ which, acting horizontally, will sustain the weight $W$ upon a smooth plane inclined at the angle $\alpha$ to the horizontal. Let the weight act at the point $A$, Fig. 20; then the force $W$ is represented by a line drawn vertically downward from $A$. The only force except $W$ and $P$ acting upon the body is the resistance of the plane, which, because the plane is smooth, acts in a normal to the plane. This is represented as in the diagram by a line $R$ drawn from the plane, because it *prevents* motion in the opposite direction. Since there are but three forces in equilibrium, their lines of action will lie in one plane, Art. 53. Accordingly, $P$'s line of action must lie in the vertical plane which is perpendicular to the inclined plane. The diagram is of course supposed to be in this plane.

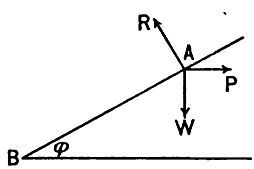


FIG. 20.

Now, since the forces represented in the diagram are in equilibrium, the resolved part of their resultant (which is zero) *in any direction*

*whatever* must vanish. That is, by Art. 61, the algebraic sum of the resolved parts of the forces along any straight line is zero, or, what is the same thing, the sum of the resolved parts in one direction along a given line is equal to the sum of those in the opposite direction. For the present purpose, let us take resolved parts along the inclined line $AB$. $R$, being perpendicular to this line, has no resolved part along it, and equating the resolved part of $P$ *up* the plane to the resolved part of $W$ *down* the plane, we have

$$P \cos \phi = W \sin \phi, \quad \ldots \ldots \quad (1)$$

whence $P = W \tan \phi$.

72. An equation formed as above by resolving forces in equilibrium along a given line is called a *condition of equilibrium*; and in equation (1) we chose the direction in such a way that the value of $R$ did not enter the equation. In like manner, if we wish to obtain $R$ directly in terms of $W$, we may resolve vertically so as not to introduce $P$; thus

$$R \cos \phi = W, \quad \ldots \ldots \quad (2)$$

whence $R = W \sec \phi$.

We may also resolve in any other direction, as for example horizontally, giving

$$P = R \sin \phi; \quad \ldots \ldots \quad (3)$$

or perpendicularly to the inclined plane, giving

$$R = P \sin \phi + W \cos \phi. \quad \ldots \ldots \quad (4)$$

Each of these equations will be found to be satisfied by the values already found for $P$ and $R$ from equations (1) and (2).

## Number of Independent Conditions.

73. The solution given in the preceding articles illustrates the fact that, in a problem where all the forces act at a single point and in a single plane, *two unknown quantities* may be determined, if all the other quantities are known, by means of two equations of equilibrium. Moreover, if two such equations are satisfied, all

other equations of equilibrium *must* be satisfied by the given and determined values. Accordingly, there are said to be but *two independent conditions of equilibrium* in such a problem, and no greater number than two unknown quantities can be determined; in other words, if more than two independent quantities are unknown, the problem is indeterminate.

When the forces are referred to rectangular axes, as in Art. 64, the two independent conditions of equilibrium in their simplest form are

$$\Sigma X = 0 \qquad \text{and} \qquad \Sigma Y = 0,$$

corresponding to a pair of components of the resultant; but we have seen that the resolutions can be made in any two convenient directions without any regard to coordinate axes.

**74.** When the forces acting on a single particle do not act in a single plane, the resolved part of the resultant in any given direction must still vanish. Equations of equilibrium are therefore found in exactly the same manner; but in this case the fulfilment of two conditions does not imply the fulfilment of all. For when the resolved parts of the forces in a single given direction vanish, all that can be inferred is that the line of action of the resultant (if there be one) is perpendicular to the given direction. Hence, when two conditions are fulfilled, corresponding to the directions of two given lines, all that we can infer is that the line of action of the resultant (if there be one) is perpendicular to the plane of the two given lines. It follows *necessarily* that the resultant can have no resolved part in any direction in the plane of the two given lines.

Hence, while a third condition is necessary to establish equilibrium, we see that resolving in a third direction in the plane will not give an *independent* condition.

But, if the resolved part of the resultant in any third direction *not* in this plane is known to vanish, the resultant itself must vanish. Hence we obtain three independent and sufficient conditions of equilibrium by *resolving forces in three directions not in the same plane.* It follows that, in a problem of forces acting at a single

point, three and not more than three unknown quantities can be determined.

When the forces are referred to rectangular coördinate axes, the three independent conditions in their simplest form are

$$\Sigma X = 0, \qquad \Sigma Y = 0 \qquad \text{and} \qquad \Sigma Z = 0^*.$$

## Solution by Means of a Triangle of Forces.

75. When there are but three forces acting, the data of the problem may be such that a triangle of forces for the equilibrium of the particle occurs in the diagram. When this is the case, the ratios of the sides of this triangle generally give the most convenient conditions of equilibrium.

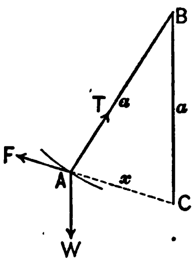

FIG. 21.

For example, let a particle at $A$ of weight $W$ be attached by a string of length $a$ to the fixed point $B$, and let $C$, vertically below $B$ at the distance $a$, be a centre of electrical repulsion of which the intensity varies inversely as the square of the distance of the particle from $C$: it is required to find the distance $AC$ when the particle is in equilibrium.

Denoting the unknown distance $AC$ by $x$, and the intensity of the repulsive force at the distance unity by $\mu$, we have

$$F = \frac{\mu}{x^2},$$

The forces acting upon the particle are its weight acting vertically, the tension of the string acting in its own direction, and $F$ acting in $AC$ produced, as indicated in the diagram. Now *because $BC$ is in this example vertical*, it follows that $ABC$ is a triangle of forces for the equilibrium of the particle at $A$. We have therefore

$$\frac{F}{x} = \frac{W}{a} = \frac{T}{a},$$

* If oblique axes were employed we should have conditions of the same form in which, however, the forces summed are not resolved parts, but components. Compare Arts. 63 and 64.

or, substituting the expression for $F$,

$$\frac{\mu}{x^3} = \frac{W}{a} = \frac{T}{a}.$$

These two independent equations determine the two unknown quantities $x$ and $T$, namely,

$$x = \sqrt[3]{\frac{a\mu}{W}} \qquad \text{and} \qquad T = W.$$

## The Condition of Equilibrium in a Plane Curve.

**76.** In the problem solved above, the particle at $A$ is restricted by the given conditions to lie in a vertical circle whose centre is $B$ and radius $a$. The problem will in fact be unchanged if we substitute for the string $AB$ a fixed *smooth* circle in this position, upon which $A$ is free to move, as a bead upon a wire. The resistance of this fixed circle, which is normal to it, will take the place of the tension $T$. In a problem of this kind, one of the unknown quantities is that which determines the *position* of equilibrium, as $x$ in the example above, and the single equation which contains this quantity and is free from the unknown resistance (or force which constrains the particle to remain in the curve) is called *the condition of equilibrium in the curve.*

When the method of resolution of forces is employed; this condition is obtained by resolving *along the tangent to the curve*, because this direction is perpendicular to that of the force of resistance. Compare Art. 72.

**77.** When the curve is given by means of its equation referred to rectangular axes, let $X$ and $Y$ denote the sum of the resolved parts, in the directions of the axes, of all the forces except the unknown resistance. Then, if $\phi$ is the inclination of the tangent at the point $(x, y)$ of the curve, the resolved parts of $X$ and $Y$ respectively along the tangent are $X \cos \phi$ and $Y \sin \phi$; and, since $X$ and $Y$ are together equivalent to all the forces acting except the resistance,

$$X \cos \phi + Y \sin \phi = 0$$

is the condition of equilibrium. Since

$$\tan\phi = \frac{dy}{dx}, \qquad \cos\phi = \frac{dx}{ds}, \qquad \sin\phi = \frac{dy}{ds},$$

this may be written

$$Xdx + Ydy = 0,$$

in which the ratio $dy : dx$ is to be derived from the equation of the curve. The result is an equation between $x$ and $y$ which, with the equation of the curve, determines the position of the point $(x, y)$ of equilibrium.

78. For example, suppose that a particle restricted to a smooth ellipse is acted upon by two forces, one toward each focus, and each varying directly as the distance; required the position of equilibrium.

The equation of the ellipse referred to its axes, as in Fig. 22, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the distance of either focus from the centre, $OF_1$ or $OF_2$, is $ae$, where $e$ is the eccentricity of the ellipse. Let $A$ be the particle in equilibrium and $P_1$, $P_2$ the forces directed along the lines $AF_1$ and $AF_2$. Draw the ordinate $AB$, then the force $P_1$ and its components, $X_1$ and $Y_1$, are proportional to the sides of the triangle $F_1AB$. Since the force $P_1$ is proportional to $AF_1$ or $r_1$, for different positions of $A$, we put $P_1 = \mu_1 r_1$, $\mu_1$ being the intensity of the force at a unit's distance. We have then

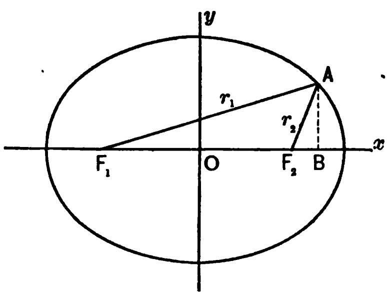

FIG. 22.

$$P_1 = \mu_1 r_1, \qquad X_1 = \mu_1 (x + ae), \qquad Y_1 = \mu_1 y;$$

and, in like manner,

$$P_2 = \mu_2 r_2, \qquad X_2 = \mu_2(x - ae), \qquad Y_2 = \mu_2 y.$$

We have then for the sums of the forces in the directions of the axes

$$X = (\mu_1 + \mu_2)x + (\mu_1 - \mu_2)ae, \quad . \quad . \quad . \quad . \quad (2)$$

$$Y = (\mu_1 + \mu_2)y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Differentiating equation (1), we obtain

$$dy = -\frac{b^2 x}{a^2 y}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in the condition of equilibrium $Xdx + Ydy = 0$, we find

$$a^2 y[(\mu_1 + \mu_2)x + (\mu_1 - \mu_2)ae] - b^2 x(\mu_1 + \mu_2)y = 0.$$

One solution of this equation is $y = 0$, which shows that each extremity of the major axis is a position of equilibrium. The other solution gives

$$x = \frac{a}{e}\,\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}.$$

When this is numerically less than $a$, it determines two intermediate positions of equilibrium.

### Condition of Equilibrium on a Fixed Curve in Space.

79. If the curve upon which the particle is constrained to lie is not a plane curve, it may be referred to three rectangular axes in space. Then, if $s$ be the length of the arc as measured from some fixed point to the point $(x, y, z)$, and $\lambda$, $\mu$, $\nu$ the direction-angles of the tangent at $(x, y, z)$, we shall have

$$\cos\lambda = \frac{dx}{ds}, \qquad \cos\mu = \frac{dy}{ds}, \qquad \cos\nu = \frac{dz}{ds}.$$

Now, if $X$, $Y$ and $Z$ denote the sums of the resolved forces in the directions of the axes, the particle is in equilibrium under the

action of the forces $X$, $Y$, $Z$ and $R$ the resistance of the curve. Since this resistance acts in some line perpendicular to the tangent, we shall obtain an equation independent of $R$ by resolving along the tangent; namely,

$$X \cos \lambda + Y \cos \mu + Z \cos \nu = 0,$$

or, substituting the values of the direction cosines,

$$Xdx + Ydy + Zdz = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation, the ratios of $dx$, $dy$ and $dz$ are those which result from the differentiation of the two relations between $x$, $y$ and $z$, which define the line to which the particle is restricted. As in the case of the plane curve, we have thus a single condition of equilibrium.

## Conditions of Equilibrium on a Surface.

**80.** Let us next suppose that the point is only restricted to lie in a given surface, of which the equation is

$$u = f(x, y, z) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Then it is plain that, if the particle at $(x, y, z)$ is in equilibrium, it would be in equilibrium if it were restricted to any line which could be drawn on the surface through the point $(x, y, z)$. It must therefore satisfy a condition of the form

$$Xdx + Ydy + Zdz = 0, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for every direction in which the point can move on the surface. When a point moves upon the surface (1), the differentials $dx$, $dy$ and $dz$ must satisfy the differential equation

$$\frac{du}{dx}dx + \frac{du}{dy}dy + \frac{du}{dz}dz = 0, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$\frac{du}{dx}, \quad \frac{du}{dy}, \quad \frac{du}{dz}$$

are the partial derivatives of the function $f$.

Comparing equations (2) and (3), which must each hold *for all values of the ratios* of $dx$, $dy$ and $dz$, we see that $X$, $Y$ and $Z$ must be proportional to the partial derivatives, that is,

$$\frac{X}{\frac{du}{dx}} = \frac{Y}{\frac{du}{dy}} = \frac{Z}{\frac{du}{dz}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This expresses the *two independent conditions of equilibrium* which must hold for a particle subject only to the single restriction of lying upon a given surface.*

## Equilibrium of Interacting Particles.

**81.** When a mutual action exists between two particles in equilibrium, the intensity of this action (which, by the third Law of Motion, is the same for each particle) is one of the forces to be considered when the conditions of equilibrium are applied to the particles separately.

For example, suppose two bodies whose weights are $P$ and $Q$ to rest at $A$ and $B$, Fig. 23, upon two planes perpendicular to one another, which intersect in a horizontal line (perpendicular to the plane of the diagram at $C$). The bodies are held apart by a rod $AB$ of fixed length; required the position of equilibrium. Denote the given inclination of the plane $AC$ by $\alpha$, and by $\theta$ the angle $BAC$, which, when found, will determine the position of equilibrium. The forces which act on either particle are, as

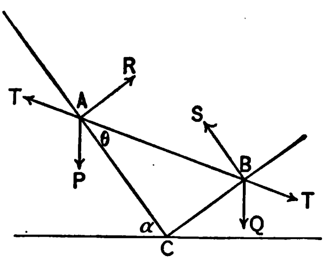

FIG. 23.

* When the given surface is the boundary of a solid substance which the particle cannot penetrate, the resistance can only act outward; but, in the general case, we suppose that the resistance may change sign; as, for example, when a particle is constrained to a spherical surface by means of a rod connecting it with a fixed point, the rod being capable of resisting either compression or tension.

represented in the diagram, its weight, the resistance of the plane on which it lies, and the thrust $T$ of the rod $AB$ (which is in compression), acting upon each in its proper direction. In the complete solution of the problem there are four equations (two for the equilibrium of each particle) and four unknown quantities, namely, the resistances $R$ and $S$, the thrust $T$, and the angle $\theta$. But, when required only to find the angle $\theta$, we may avoid the resistances and obtain two equations for the remaining quantities by resolving in each case along the plane only. Thus we obtain

$$P \sin \alpha = T \cos \theta,$$

and

$$Q \cos \alpha = T \sin \theta;$$

whence, eliminating $T$, we have

$$\tan \theta = \frac{Q}{P} \cot \alpha,$$

which determines the value of $\theta$.

## EXAMPLES. IV.

1. Show that three forces represented in magnitude and direction by the medial lines of a triangle, and acting at a point, are in equilibrium.

2. A weight of 25 pounds hangs by two strings, of which the lengths are 3 and 4 feet respectively, from two points in a horizontal line, distant 5 feet from each other. Find the tension of each string. 20 pounds; 15 pounds.

3. A body is sustained upon an inclined plane by two forces, each equal to half the weight, one horizontal and the other acting along the plane. Determine the inclination of the plane.

$\phi = 2 \tan^{-1}\frac{1}{3}$.

4. A weight of 50 pounds, moving in smooth vertical guides is supported by a string, attached to a point at a horizontal distance of 12 feet from the guides and 5 feet above the weight. Find the tension of the string and the pressure on the guides.

130 pounds; 120 pounds.

5. Two weights, $P$ and $Q$, are attached to the extremities of a string which passes over a smooth peg at a distance $b$ vertically over the centre of a sphere of radius $a$, on whose surface $P$ rests while $Q$ hangs freely. Find the distance $s$ of $P$ from the peg when in equilibrium.

$$s = \frac{Qb}{P}.$$

6. A picture, whose weight is $W$, hangs from a nail by means of a cord, whose length is $l$, attached to two screw-eyes the horizontal projection of whose distance is $a$. What is the tension of the cord

$$\frac{Wl}{2\sqrt{(l^2-a^2)}}.$$

7. Weights $P$ and $Q$ are attached to the extremities of a cord of length $l$, which passes over a smooth peg at the distance $h$ vertically above the centre of a smooth sphere of radius $a$, upon whose surface $P$ and $Q$ rest in equilibrium. Show that the cord is divided at the peg into segments inversely proportional to the weights, and find the tension of the cord.

$$\frac{PQl}{(P+Q)h}.$$

8. An anchor weighing 4000 pounds is supported by two tackles from the fore and main yards of a vessel, making angles of 30° and 45° respectively with the vertical. Find the tension on each tackle. 2928 and 2070 lbs.

9. A weight $W$ is supported by a tripod each leg of which is $3\frac{1}{2}$ feet long, the feet making a triangle each side of which is $2\frac{1}{2}$ feet long. Find the thrust in each leg.

$$\frac{14W}{39}.$$

10. Two smooth rings of weights $P$ and $Q$ rest on the convex side of a circular wire in a vertical plane, and are connected by a string subtending the angle $2\alpha$ at the centre. Determine the inclination $\theta$ of the string to the vertical in the position of equilibrium.

$$\tan\theta = \frac{P+Q}{P-Q}\cot\alpha.$$

11. Two smooth pegs are in the same horizontal line and 6 feet distant. The end of a string is made fast to one of them, and passing over the other sustains a weight of 10

pounds, while a smooth ring weighing 13 pounds is suspended on the bight. Find the length of string between the pegs.

$$\frac{120}{\sqrt{231}}\text{feet.}$$

12. A weight $W$ is attached to a ring $A$ which slides on a smooth circular hoop in a vertical plane. An equal weight is also attached to the ring by means of a string which passes over a smooth peg at the extremity $B$ of a horizontal diameter of the hoop. Find the position of $A$. 120° from $B$.

13. A barrel four feet long, weight 500 lbs., is hoisted from a ship's hold by means of a pair of can-hooks 52 inches long. Find the tension on each leg of the can-hook. 650 lbs.

14. A body is kept in equilibrium on a smooth plane of inclination $\alpha$ by a force $P$ acting along the plane and a horizontal force $Q$. When the inclination is halved and the forces $P$ and $Q$ each halved, the body is observed to be still in equilibrium. Find the ratio of $P$ to $Q$.

$$\frac{P}{Q} = 2\cos^2\frac{\alpha}{4}.$$

15. A ring of weight $W$ slides on a smooth rod fixed at an inclination of 30° to the horizontal. A weight $W'$ is attached to one end of a string which passes through the ring, the other end being attached to a fixed point not in the rod. Prove that there is no position of equilibrium unless $W < W'$.

16. Two spheres of radii $a$ and $b$ and weights $W$ and $W'$ (which are supposed to act at the centres) are connected by a string of length $l$ attached to points in their surfaces. They are in equilibrium when hung by the string over a smooth peg, their surfaces being in contact. Find the parts into which the peg divides the string.

$$\frac{W'l + W'b - Wa}{W + W'} \text{ and } \frac{Wl + Wa - W'b}{W + W'}.$$

17. A bead is movable on a circular wire whose plane is vertical; a string attached to it passes through a smooth ring at the highest point of the circle and supports a weight at its other end equal to that of the bead. Find the angle between the parts of the string in the position of equilibrium. 60°.

18. The ends of a string are attached to two heavy rings of weight $W$ and $W'$ which are free to slide upon two smooth fixed rods making the angles $\alpha$ and $\beta$ with the horizontal and in the same vertical plane; the string carries a third ring of weight $M$ which slides upon it. Prove that, if $\phi$ is the angle which each part of the string makes with the vertical,

$$\cot\phi : \cot\beta : \cot\alpha = M : M + 2W' : M + 2W$$

# CHAPTER III.

## FORCES ACTING IN A SINGLE PLANE.

### V.

### Joint Action of Forces on a Rigid Body.

82. When forces act upon a solid body, the points at which the forces are applied are of importance in considering the *motion* produced. But in *statics*, the present division of our subject, we are concerned only with the *tendency* to motion, or change of motion, while the body is in a single definite position. The principle of transmission of force shows that no change of action is produced if the point of application is transferred to any point of the line of action, provided the new point of application is a point of the same rigid body—in other words, *rigidly connected* with the original point of application. It follows that it is *the position of the line of action* only, and not that of the point of application, which is at present of consequence. When however, we come to treat of the motion of the body the latter will be of consequence, because it affects the position of the line of action.

We confine ourselves in this chapter to the action of forces whose lines of action lie in one plane, and we shall find that the joint action of such forces is in general the same as that of a certain single force which is called their resultant. The body upon which the forces are supposed to act is frequently not represented in the figure at all; but if there is a resultant force, it is of course assumed that some point of its line of action is rigidly

connected with the supposed body, so that it might serve as the point of application.

## Construction of the Resultant.

83. Let $P_1$, $P_2$ and $P_3$, Fig. 24, represent in magnitudes and lines of action three forces acting in a plane. Let the lines of action of $P_1$ and $P_2$ intersect at $A$; these forces may be transferred to $A$ and their resultant $Q$ constructed at that point. The joint action of $P_1$, $P_2$ and $P_3$ is evidently the same as that of $Q$ and $P_3$. Hence, constructing, in like manner, the resultant of $Q$ and $P_3$ at $B$, the point of intersection of their lines of action, we have a force $R$ whose action is the same as the joint action of $P_1$, $P_2$ and $P_3$. In like manner, we may construct the resultant of any number of forces in a plane, provided that at no step the forces to be combined have parallel lines of action.

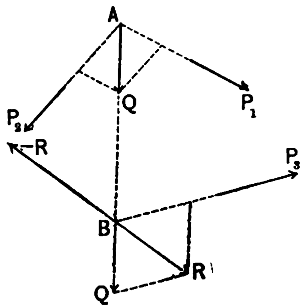

FIG. 24.

It is evident that, in this construction, the magnitude and direction of the resultant are the same as if the forces, retaining their magnitudes and directions, had all acted at a single point. Thus the resultant *considered only as a vector* may be found from the given forces by simple vectorial addition as in Art. 58. But, in the present construction, we have in addition found the position of the resultant line of action.

84. The construction is simplified by separating completely the determination of the resultant vector from that of the line of action, as illustrated in Fig. 25.

Let $P_1$, $P_2$, $P_3$ and $P_4$ be four forces given in magnitude, direction and position, to find the resultant. Taking any point $O$ as origin, we first construct the resultant vector by laying off from $O$ successively $OA$ equal and parallel to $P_1$, $AB$ equal and parallel to $P_2$, $BC$ to $P_3$, and $CD$ to $P_4$. Then, as in Art. 58, $OD$ is the vector representing the resultant. Now, to find the position

of the line of action, join $OB$, $OC$. Through the intersections of the lines of action of $P_1$ and $P_2$, whose vectors were used in finding $B$, draw a line parallel to $OB$: the resultant of $P_1$ and $P_2$ is the force $OB$ acting in this line. Next, through the intersection of this line with the line of action of $P_3$ draw a parallel to $OC$; the resultant of $P_1$, $P_2$ and $P_3$ is the force $OC$ acting in this line. Proceeding in this way we finally arrive at the line in which acts the required resultant $R$, which is equal and parallel to $OD$, as represented in the diagram.

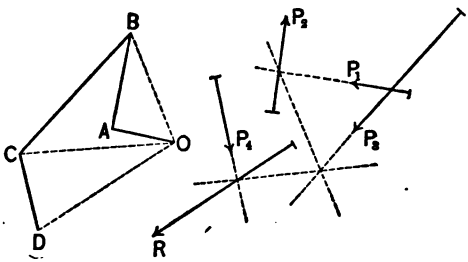


FIG. 25.

If any change is made in the order of the forces, we shall arrive in the first part of the construction at the same point $D$, and in the second part, at some final point of intersection which will determine the same line of action for $R$. Such a new construction may be used to test the accuracy of the drawing.

## The Resultant of Two Parallel Forces.

**85.** The method given in Art. 83 fails when the two forces to be combined act in parallel lines, because there is no point of intersection to which we can transfer them. The difficulty is obviated by introducing two equal and opposite forces in any line of action which intersects the parallel lines. Thus, in Fig. 26, let the forces $P$ and $Q$ act in parallel lines at $A$ and $B$. To find their resultant, let two forces, each equal to $F$, acting in opposite directions in the line $AB$, be introduced. Since these last forces counterbalance each other, it is plain that the system of four forces acting at $A$ and $B$ will have the same

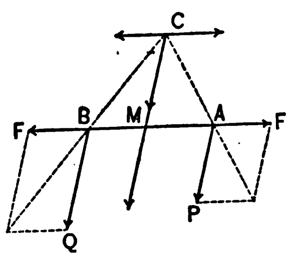


FIG. 26.

resultant as $P$ and $Q$. Constructing now the resultants of the pair of forces at $A$ and the pair at $B$, we have two forces whose resultant is the same as that of $P$ and $Q$. Their lines of action meet at $C$, which is therefore on the line of action of the final resultant. If we now again resolve the two forces at $C$ into components equal and parallel to their original ones, we shall have the system replaced by four forces acting at $C$; namely, two forces equal to $F$, which counterbalance each other, and the forces $P$ and $Q$, of which the resultant is the force $P+Q$ acting at $C$.

**86.** As before, the resultant, regarded as a vector, is the same as if the forces acted at a single point; but we have also determined the position of the line of action. Let this line intersect $AB$ in $M$; then the triangles $CMA$ and $CMB$ are similar to those used in the construction; therefore

$$\frac{CM}{MA}=\frac{P}{F} \quad \text{and} \quad \frac{CM}{MB}=\frac{Q}{F}.$$

Dividing,

$$\frac{MB}{MA}=\frac{P}{Q}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

hence the point $M$ divides the line $AB$ *inversely* in the ratio of the forces. Since $A$ and $B$ are any points in the lines of action of $P$ and $Q$, we see that the lines of action of two parallel forces $P$ and $Q$ and their resultant $R=P+Q$ cut any transverse line in points $A$, $B$, and $M$ such that

$$P:Q:R=BM:MA:BA, \quad . \quad . \quad . \quad . \quad (2)$$

each force being proportional to the distance between the lines of action of the other two. We have seen in Art. 50 that, in the general case, each force is proportional to the sine of the angle between the lines of action of the other two. The present proposition is in fact the limiting case of the former.

**87.** Conversely, a force $R$ may be resolved into components acting in any two lines parallel to its line of action and in one

plane with it. Thus if, in Fig. 26, the force $R$ is given in position and magnitude, and the given lines of action of $P$ and $Q$ are on opposite sides of it, we have from equations (2)

$$P = \frac{BM}{BA}R \quad \text{and} \quad Q = \frac{MA}{BA}R, \quad . \quad . \quad . \quad (3)$$

which determine the magnitudes of the components.

**88.** If the two given parallel forces act in opposite directions, the construction is the same, but the point $C$ falls beyond the line of action of the greater force $P$. The demonstration and deduction of equations (1) and (2) are precisely the same as in the preceding case; but the resultant is in the direction of the larger force $P$, and equal to $P-Q$. The line $AB$ is said to be divided *externally* in the inverse ratio of the forces. So also, in the converse problem, where $R$ is given to be resolved into two components in given parallel lines of action both on the same side of the line of action of $R$, equations (3) of the preceding article hold for the magnitudes of the components; but the component nearer to $R$ exceeds it in magnitude, while the more distant is in the opposite direction.

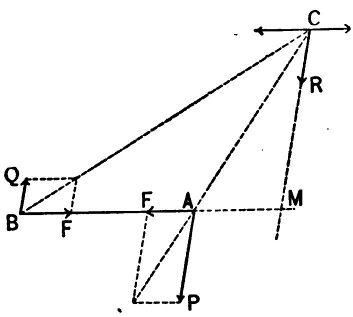


FIG. 27.

## The Resultant of a Number of Parallel Forces in One Plane.

**89.** It is an obvious consequence of the preceding articles that, for any number of parallel forces, the resultant considered as a vector, or *resultant force*, is the algebraic sum of the given forces. But, in finding the position of the line of action graphically, it is more convenient, instead of combining the parallel forces two by two, to use a process similar to that of Art. 84. Thus, in Fig. 28, let forces $P_1$, $P_2$, $P_3$ and $P_4$ act in the parallel

lines as represented. Assume also a force $Q$, acting in a line intersecting the line of action of $P_1$. Taking any point $O$, we construct vectorially the resultant of $Q$, $P_1$, $P_2$, $P_3$ and $P_4$ by laying off $OA$ equal and parallel to $Q$, and then $AB$, $BC$, $CD$, $DE$ equal and parallel to the given forces. Completing the figure, as in Art. 84, $OE$ represents vectorially the resultant of $Q$ and the given forces. To find the line of action of this resultant, draw through the intersection of

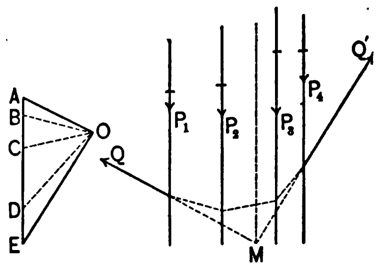


FIG. 28.

the lines of action of $Q$ and $P_1$ a parallel to $OB$; the resultant of these two forces is therefore $OB$ acting in this line. Again, through the intersection of this line with the line of action of $P_2$, draw a line parallel to $OC$: the resultant of $Q$, $P_1$ and $P_2$ is the force $OC$ acting in this line. Continuing in this manner, we finally obtain the line of action of the resultant $OE$. Now the resultant of this force with one equal and opposite to $Q$ in the same line of action, is the resultant of the parallel forces; hence the resultant line of action passes through $M$, the intersection of this last line with the line of action of $Q$. Thus the resultant of the parallel forces is the force $R$ equal to their algebraic sum acting in a parallel line through $M$.

90. If we reverse the direction of the resultant force $OE$, and denote it (in the direction $EO$) by $Q'$, we have six forces, namely, $Q$, $P_1$, $P_2$, $P_3$, $P_4$ and $Q'$, in equilibrium, as represented vectorially by the closed polygon $OABCDEO$ in the left-hand figure, which is called the *force diagram*. The resultant of $Q$ and $Q'$ is therefore the reverse of the resultant of the parallel forces. Supposing the parallel forces to be the weights of given bodies acting in vertical lines, $Q$ and $Q'$ will be oblique forces which are together capable of sustaining the weights while acting in the given lines. The broken line formed in the construction of the right-hand figure may be regarded as a cord to which the given

weights are knotted at the points of intersection, and of which the extremities are attached to fixed points in the lines of action of $Q$ and $Q'$. The figure is hence called a *funicular polygon* for the given parallel forces.

Taking into consideration the action and reaction at its two ends of the tension in each of the intermediate segments, we notice that the triangles in the force diagram represent the separate equilibrium of each of the knots. The term funicular polygon is sometimes extended to the more general case, in which the points of application of the forces are supposed connected by rods, some of which may be in compression instead of tension. Compare Arts. 122 and 125.

## Couples.

**91.** When the parallel and oppositely directed forces in Art. 88 are equal, the construction fails because the lines by which the point $C$ was found are, in this case, parallel. Indeed the magnitude of the resulting force *considered as a vector* is now zero, and yet the forces are not in equilibrium. This combination of two equal opposite forces acting in parallel lines is called a *couple* because it cannot be reduced to any simpler mechanical equivalent. Thus Fig. 29 represents a couple acting upon a rigid body, which may here be regarded as a *lamina* or thin plate in the plane of the two parallel lines of action. The mechanical action of the couple is obviously a tendency to turn the lamina in its own plane. This tendency is called a *turning moment* or simply a *moment*. It cannot be counteracted by means of a single force, and if motion is prevented by the resistance of fixed bodies in contact with that on which the couple acts, the reaction of these bodies is equivalent to a turning moment in the opposite direction of rotation.

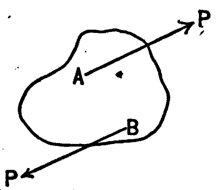


FIG. 29.

## Measure of Turning Moment.

**92.** We have seen in Art. 86 that the resultant of two parallel forces in the same direction acts in a line dividing a transverse line in the inverse ratio of the forces. A force opposite to this resultant will therefore produce equilibrium. Thus, in Fig. 30, in which the transverse line $AB$ is taken perpendicular to the parallel lines of action, the force $P+Q$ acting at $M$ is in equilibrium with $P$ and $Q$, if

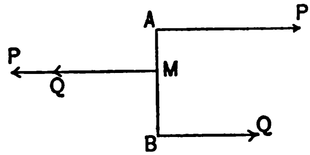


FIG. 30.

$$AM : MB = Q : P, \quad \ldots \ldots \quad (1)$$

or

$$P \times AM = Q \times MB. \quad \ldots \quad (2)$$

Distinguishing the direction of $P$ and $Q$ as the positive direction along the parallel lines, the forces in equilibrium may be regarded as forming two couples; namely, $P$ acting at $A$ with $-P$ acting at $M$, and $Q$ acting at $B$ with $-Q$ acting at $M$. These two couples are therefore in equilibrium; in other words, their turning moments, which are in opposite directions, are equal in magnitude. Equation (2) shows that *the product of the magnitude of the force and the distance between the parallel lines of action* is the same for each of these couples. This product is therefore taken as *the measure of the moment* of the couple. For example, if $Q = 2$ pounds and $BM = 3$ feet, the moment of the $Q$-couple is said to be 6 pounds-feet, and when algebraic signs are used this moment is taken as positive because it tends to produce positive rotation.

## Moment of a Force about a Point.

**93.** If the point $B$ of the lamina in Fig. 29 is fixed, and the lamina, while free to turn about $B$, is acted upon by a force $P$ at the point $A$, it will tend to turn about the point $B$. The measure of this tendency is called *the moment of the force about the point B.* The resistance at $B$ which prevents the motion of

that point of the lamina is a force equal, parallel and opposite to $P$, and the turning effect is produced by the couple thus formed. Therefore its measure is taken to be the same as that of the couple; that is, *the moment of a force about a given point is the product of the magnitude of the force and the perpendicular from the point upon the line of action.*

Accordingly in Fig. 30 the forces $P$ and $Q$ are said to have equal and opposite moments about $M$; and if $AB$ is a rigid bar free to turn about a fixed point $M$, it will be in equilibrium when parallel forces $P$ and $Q$ act at $A$ and $B$, provided $M$ divides $AB$ in the inverse ratio of the forces. In this arrangement, the bar is called a *lever*, the point $M$ *the fulcrum*, and $AM$, $MB$ *the arms.*

The proposition just stated, upon which is based the measure of moments, has been known from the early days of mechanical science as *the principle of the lever.* The perpendicular from the point is often called *the arm* of the moment so that the moment is said to be *the product of the force and the arm.* In like manner the distance between the parallel lines of action is called *the arm of the couple.*

**94.** It is sometimes convenient to employ, as the line factor in the expression for a moment, the distance of a definite point of application of the force from the point about which the moment is taken. This can be done by means of the following theorem: *The moment about O of a force P acting at A is equal to the product of OA and the resolved part of P in the direction perpendicular to OA.*

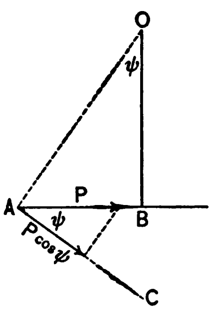


FIG. 31.

To prove this, let $AC$, Fig. 31, be perpendicular to $OA$, and $OB$ perpendicular to the line of action; then, denoting the angle $BAC$ by $\psi$, we have $OB = OA \cos\psi$, and the resolved part of $P$ in the direction $AC$ is $P\cos\psi$. Hence, denoting the moment by $H$, we have

$$H = P \times OB = P \times OA \cos\psi = P\cos\psi \, . \, OA,$$

that is, the product of $OA$ by the resolved force. Since $OA$ is the ***arm*** for the resolved force, we may state the theorem thus: *The moment about O of a force acting at A is the same as the moment of its resolved part perpendicular to OA.*

## Varignon's Theorem of Moments.

**95.** Varignon's theorem, that: *The moment about any point of the resultant of two forces is the algebraic sum of the moments of the given forces,* follows directly from the theorem of the preceding article. Thus, in Fig. 32, let $O$ be the point about which the moment is taken, or *origin of moments,* and let $P$ and $Q$ be the forces acting in lines which intersect at $A$. Construct the resultant $R$ at $A$, join $OA$, and draw $AB$ at right angles to $OA$. Resolving forces in the direction $AB$ by projecting the lines representing $P$, $Q$ and $R$ upon $AB$, we have

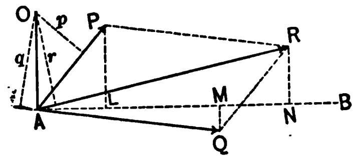

FIG. 32.

$$AN = AL + AM.$$

Multiplying by $OA$, we derive the equation

$$OA \,.\, AN = OA \,.\, AL + OA \,.\, AM,$$

in which, by Art. 94, the several terms are the moments of $R$, $P$ and $Q$ about $O$. Hence, denoting the perpendiculars from $O$ by $r$, $p$ and $q$,

$$rR = pP + qQ.$$

In the diagram, $AB$ is taken as the positive direction for the resolved forces, because a force in that direction has a positive moment about $O$. Then the signs of the moments are the same as those of the resolved forces; and, for all positions of $O$, we have the moment of the resultant equal to the algebraic sum of the moments of the given forces.

**96.** When the forces are parallel, let a perpendicular be

drawn, as in Fig. 33, from $O$, cutting the parallel lines of action in $A$, $B$ and $C$, at which points we may regard $P$, $Q$ and $R$ as acting. Then we have seen in Art. 86 that

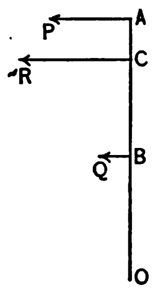


FIG. 33.

$$P \,.\, AC = Q \,.\, BC. \quad . \quad . \quad . \quad (1)$$

Denote the distance $OC$ by $x$; then the sum of the moments of $P$ and $Q$ about $O$ is

$$P(x + AC) + Q(x - BC),$$

which by equation (1) reduces to

$$(P + Q)x.$$

But this is the moment of $R$ about $O$, since $R = P + Q$, and $x$ is the arm $OC$. Hence, as before, the moment of the resultant is the sum of the moments of the components.

The proof is readily extended to cases in which $O$ is between the lines of action, and to that in which the forces have opposite directions. Thus, in Fig. 34, we have as before

$$P \,.\, AC = Q \,.\, BC,$$

and the algebraic sum of the moments is

$$P(x - AC) - Q(x - BC),$$

which reduces to $(P - Q)x$ or $Rx$.

Thus Varignon's Theorem is true for any parallel and unequal forces.

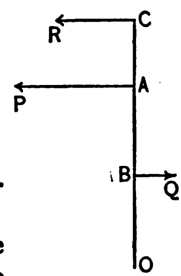


FIG. 34.

97. Finally, when the two given forces are parallel, equal and opposite, their resultant is the couple which they form, and the theorem is that: *The algebraic sum of the moments of the forces forming a couple has for every point in the plane of the couple the same value as the moment of the couple.* To prove this, draw through $O$ a line $OBA$ perpendicular to the parallel lines of action, as in Fig. 35. Supposing $O$ to be beyond the lines of action as indicated, denote its distance $OB$ from the nearer line by $x$, and the arm $AB$ of the couple by $a$. Then the moment about $O$ of $P$ acting at $A$ is $P(a + x)$, and that of $P$ acting at $B$ is

$Px$ in the opposite direction; hence the algebraic sum of the moment is

$$P(a + x) - Px = Pa,$$

which is independent of $x$, and has been already defined in Art. 92 as the moment of the couple. When the point $O$ is between the lines of action the moments of the components have the same algebraic sign, and the moment of the couple is their numerical sum.

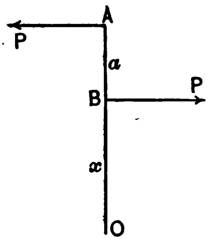


FIG. 35.

98. If $H_1$, $H_2$ and $H_3$ denote the moments of the forces $P_1$, $P_2$ and $P_3$ about a selected point $O$ in the plane of the forces, and $Q$ the resultant of $P_1$ and $P_2$, the moment of $Q$ about $O$ is $H_1 + H_2$ by Varignon's Theorem. Again, if $R$ is the resultant of $Q$ and $P_3$, its moment about $O$ is, by the same theorem, $H_1 + H_2 + H_3$. But $R$ is the resultant of $P_1$, $P_2$ and $P_3$; hence the moment of the resultant of these forces is equal to the algebraic sum of the moments of the forces; and, in like manner, for any number of forces, if $K$ denote the moment of the resultant $R$, we have

$$K = H_1 + H_2 + \ldots + H_n = \Sigma H.$$

This resultant moment of a system of forces in a plane is frequently called simply the moment of the system with respect to the given origin of moments.

## Three Numerical Elements Determining a Force in a Given Plane.

99. A force acting in a given plane and at a given point, or considered merely as a vector in a given plane, requires two numerical values for its determination. These may be the values of $P$ and $\theta$, the magnitude and one angle determining the direction of the force; but we have seen in Art. 63 that the most convenient determining elements (or *coordinates*, in the general sense of the term) are the values of $X$ and $Y$, the components of the force in two standard directions, because these are combined in the resultant by simple algebraic addition.

When the forces in the given plane are not confined to a single point of application, they are not completely represented by vectors, and we require in addition a third numerical element to fix *the position of the line of action* after its direction has been fixed. This third element might be taken as the perpendicular from a fixed point of reference upon the line of action; but the theorem of moments shows that it is more convenient to take for the third element the moment about the point of reference, because then the values of this element also are combined in the resultant by simple algebraic addition.

We therefore take $X$, $Y$ and $H$ (the resolved forces in two fixed directions and the moment about a fixed point) for the three determining elements of a force $P$ in a given plane; then

$$\Sigma X, \quad \Sigma Y \quad \text{and} \quad \Sigma H$$

are the corresponding determining elements of the resultant of a system of forces in one plane.

**100.** Let $R$ denote the resultant vector of the given system of forces, $K$, as in Art. 98, the resultant moment, and $p$ the perpendicular from $O$ the origin of moments upon the line of action of the resultant. Then, by the theorem of moments,

$$K = Rp,$$

which determines the value of $p$. The direction of the vector $R$ determines that of the perpendicular line upon which $p$ is to be laid off, and the sign of $K$ determines in which of the two opposite directions it is to be laid off from $O$.

It is to be noticed that, in the case of the single force, if we have $X = 0$ and $Y = 0$, we shall have $P = 0$, and therefore $H = 0$ for any position of the origin of moments $O$. But in the case of the resultant, we may have $\Sigma X = 0$ and $\Sigma Y = 0$, (whence $R = 0$,) without having $K = 0$. In this last case, the resultant is not a force, but the couple whose moment is $K$; and the value of $K$ is, in this case, independent of the position of $O$.

The system of forces will produce equilibrium only when all three of the elements vanish, that is, when $\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma H = 0$.

## Resultant of a Force and a Couple.

**101.** Since a couple in a given plane has no element of magnitude except that of moment, the direction and magnitude of the force $P$, employed in the graphical representation of a couple, are immaterial, provided only the arm $a$ be so taken that $aP = H$, the given value of the moment of the couple. Hence, to find the resultant of a force $P$ acting at $A$, Fig. 36, and the couple $H$, we may put

$$H = aP$$

(which determines $a$), and then represent $H$ by the force $P$ reversed at $A$ and an equal force acting in a parallel line at the distance $AB = a$ from the original line of action. (In the diagram $AB$ is laid off on the supposition that $H$ is positive.) Then the two forces acting at $A$ neutralize each other; therefore, the resultant of $P$ at $A$ and the couple $H$ is the force $P$ acting at $B$. Thus the result of combining a couple with a force, is not to change it as a vector, but to shift its line of action. The effect is algebraically to increase its moment with respect to any point by a constant quantity.

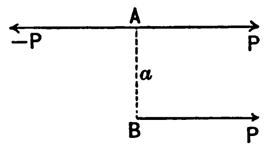

FIG. 36.

**102.** Conversely, the force $P$ acting at $B$, Fig. 36, may be resolved into an equal and parallel force, acting at any selected point $A$, and a couple; and the moment $H$ of this couple is that of the force about the selected point.

The process of combining forces into a resultant in Art. 100 may in fact be described thus: We first resolve each of the forces into a parallel force acting at the selected origin of moments $O$ and a couple $H$; we next combine the forces acting at $O$ into a resultant $R$, and the couples $H$ (by algebraic addition) into a couple $K$; finally, unless $R = 0$, we combine the couple $K$ with $R$ acting at $O$, so as to shift its line of action as in Art. 101.

## Moment of a Force Represented by an Area.

**103.** When a force $P$ is represented by a line of definite length and in a definite position, its moment about a point $O$ is represented by *twice the area of the triangle whose base is the line*

*representing $P$ and whose vertex is $O$*; for the altitude of this triangle is $p$, the arm of the moment, and the area is one half the product of the base and altitude.

As an application, let us consider a system of forces acting in the sides of a polygon in one consecutive direction around the perimeter and proportional to the sides in which they act. Thus, in Fig. 37, let the forces be represented in magnitude and position by $AB$, $BC$, $CD$, $DE$, $EA$. Take any point $O$; then, joining it with $A$, $B$, $C$, $D$ and $E$, the moments of the several forces about $O$ are represented by the doubles of the triangles $AOB$, etc. Therefore $K$, the sum of the moments (which in the diagram have all the same sign), is twice the area of the polygon. Since the *vectorial* sum of the forces is by hypothesis zero, the resultant is not a force. It is therefore a couple $K$ measured by twice the area of the polygon. Accordingly, we find here, as in Art. 97, that the resultant moment is the same for every position of the point $O$.

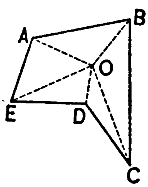

FIG. 37.

## Forces in a Plane Referred to Rectangular Axes.

**104.** When the forces and their points of application are referred to rectangular axes, the moments are usually taken about the origin. In Fig. 38, let $X$ and $Y$ be the rectangular components along the axes of the force $P$, and let $x$ and $y$ be the coordinates of its point of application $A$. The moment of $P$ about the origin is the algebraic sum of the moments about the same point of $X$ and $Y$ acting at $A$. The numerical values of these moments are $yX$ and $xY$. When $x$, $y$, $X$ and $Y$ are all positive, as in the figure, it will be noticed that the moment of $Y$ is positive and that of $X$ is negative; hence the moment of $P$ about the origin is

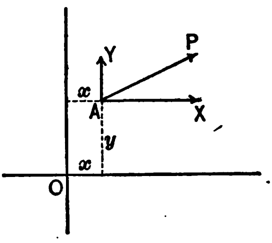

FIG. 38.

$$H = xY - yX. \quad \ldots \ldots \ldots \quad (1)$$

**105.** When the force is determined by given values of the elements $X$, $Y$ and $H$, as in Art. 99, the point of application $(x, y)$ is not fully determined; for, it has only to satisfy equation (1), which is therefore *the equation of the line of action.* We have seen, in Art. 64, how the inclination to the axis of $x$ and the value of $P$ are determined; and, denoting by $p$ the perpendicular from the origin upon the line of action, $p$ is determined by $H = Pp$.

In like manner, in the case of a system of forces, we have, for the rectangular components of the resultant and the resultant moment about the origin,

$$X' = \Sigma X, \qquad Y' = \Sigma Y, \qquad K = \Sigma xY - \Sigma yX;$$

hence the equation of the line of action of the resultant is

$$xY' - yX' = K. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Accordingly, we find that, if $K = 0$, this equation represents a line passing through the origin. If $X' = 0$, it represents a line parallel to the axis of $y$; and if $Y' = 0$, a line parallel to the axis of $x$. If $X' = 0$ and $Y' = 0$, while $K$ does not vanish, the equation becomes the impossible one representing the line at infinity, the resultant being, in this case, the couple whose moment is $K$.

## EXAMPLES. V.

1. *ABCD* is a square; a force of one pound acts along *AD*, a force of two pounds along *AB*, and a force of three pounds along *CB*. Determine the resultant and its line of action.

2. *ABCD* is a parallelogram; forces represented in magnitude and position by *AB*, *BC* and *CD* act on a body. Determine the resultant. Reversing the resultant, so as to produce equilibrium, explain the result by the theory of couples.

3. Show that a force may be graphically resolved into two components, one acting along a given line of action coplanar with that of the force, and the other at a given point in the same plane.

4. Show how to resolve a force graphically into three components acting along three given lines coplanar with the line of

action. When is this impossible? and when does one component vanish?

5. $ABCD$ is a square. A force of 3 lbs. acts from $A$ to $B$, a force of 4 lbs. from $B$ to $C$, a force of 6 lbs. from $D$ to $C$, and a force of 5 lbs. from $A$ to $D$. Show that the line of action of the resultant force is parallel to the diagonal $AC$, and find where it crosses $AD$. At a distance of $\frac{2}{3}AD$ from $A$.

6. The distance between the lines of action of two parallel forces $P$ and $Q$ is $a$. What is the moment of either force about a point in the line of action of the resultant? $\dfrac{aPQ}{P+Q}$.

7. A man supports two weights slung on the ends of a stick 40 inches long placed across his shoulder. If one weight be two thirds of the other, find the point of support, the weight of the stick being disregarded.

16 inches from the larger weight.

8. To a rod 10 feet long, carried by $A$ and $B$, a weight of 100 pounds is slung, by means of two cords; one, 4 feet long, attached to a point 2 feet from $A$'s end, the other, 3 feet long to a point 3 feet from $B$'s end. Determine the portions carried by $A$ and $B$. 48 lbs.; 52 lbs.

9. If the side of the square in Ex. 1 is two feet in length, find the resultant moment about $C$ and about $D$.

2 pounds-feet in each case, directions opposite.

*In the following problems, the weight of a uniform beam or rod is regarded as acting at its middle point.*

10. A horizontal uniform beam, 5 feet long and weighing 10 pounds, is supported at its ends on two props. How far from one prop must a weight of 30 pounds be placed on the beam, in order that the pressure on that prop may be 25 pounds?

20 inches.

11. A rod weighing $\frac{1}{2}$ pound per foot turns about a smooth hinge at one end, and is held by a string fastened to a point 10 inches from the other end. If the string can only sustain $1\frac{1}{2}$ pounds tension, find the limiting lengths of the rod, when held in a horizontal position. 1 and 5 feet.

12. A bar $AB$, weighing $\frac{1}{4}$ of a pound per linear inch, rests on a prop at $A$ and carries a weight of 125 pounds at a point 10 inches from $A$. Find the length of the bar, in order that the force $P$ acting at $B$ to produce equilibrium may be the least possible. 8 feet 4 inches.

13. The forces $P$ and $Q$ act at $A$ and $B$ perpendicularly to the arms of a bent lever, or "bell-crank," $ACB$ which turns about the fulcrum $C$. Show that in equilibrium the resultant of $P$ and $Q$ passes through $C$, and thence derive the measure of a turning moment.

14. Weights of 3 pounds and 5 pounds respectively hang from pegs in the rim of a vertical wheel, whose radius is 2 feet, at the extremity of a horizontal radius and at a point 120° distant. What is the resulting moment at the centre? and if the wheel be blocked by a fixed peg touching a spoke at a point 3 inches from the centre, what is the pressure on the peg?

1 pound-foot; 4 pounds.

15. Demonstrate the equivalence of two couples having the same moment but different forces and arms, by reversing one of them and showing that the four forces are in equilibrium.

16. $ABC$ is a triangle. Show how to construct the line of action of a force whose moments about $A$, $B$ and $C$ are in the ratios $l : m : n$.

17. Verify geometrically the value of $p$ found from the value of $H$ in Art. 104.

18. If $H_0$ is the moment at the origin, and $X$, $Y$ the resolved parts of a force referred to rectangular axes, show that

$$H = H_0 - Yx + Xy$$

expresses the moment of the force about the point $(x, y)$.

19. If four forces acting along the sides of a quadrilateral are in equilibrium, prove that the quadrilateral is plane; and that, if it can be inscribed in a circle, the forces are proportional to the opposite sides.

20. Four forces act in, and are inversely proportional to, the sides $AB$, $BC$, $CD$ and $DA$ of a quadrilateral inscribed in a circle.

Show that the resultant moment about the intersection of $AB$ and $CD$ vanishes; and, thence, that the line of action of the resultant passes through the intersections of pairs of opposite sides.

21. Four forces acting in the sides of a trapezium are in equilibrium. Prove that the forces in the non-parallel sides may be represented by the sides themselves, and those in the parallel sides each by the opposite side.

## VI.

## Conditions of Equilibrium for Forces in a Single Plane.

**106.** The forces acting upon a rigid body at rest, including the resistances of other bodies with which it is in contact, form a system in equilibrium. We consider in this section cases in which the forces act in a single plane, but at different points of application.

When such a system is in equilibrium, not only must the resultant force $R$, considered as a vector, vanish; but $K$, the resultant moment (Art. 98) of the forces about any selected point, must vanish. We therefore have, in addition to the general condition of equilibrium used in § IV (namely, that the resolved forces in the direction of any straight line must balance each other), a new general condition; namely, that *the algebraic sum of the moments of the forces about any point in the plane must vanish;* or, what is the same thing, that the sum of the moments of the forces tending to turn the body in one direction about the point must be equal to the sum of those tending to turn it in the opposite direction.

**107.** As in Art. 71, it is important to represent, in the diagram constructed for a problem, *all* the forces, including the resistances, which act upon the single body whose equilibrium is considered, and *only* those forces. As an illustration, take the following problem: Let $AB$, Fig. 39, represent a uniform heavy beam,

6 feet long, resting at $A$ upon a smooth horizontal plane, and at $D$ upon the smooth top of a vertical post, 3 feet high, fixed in the plane. The end $A$ is prevented from slipping by a cord $AC$, 4 feet long, connecting it with the foot of the post. Required the tension of this cord.

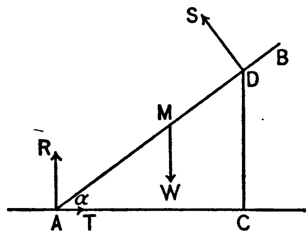


FIG. 39.

The only force acting from a distance is the weight of the beam, which, because the beam is uniform, may be regarded as acting at its middle point $M$. The remaining forces are the resistances of fixed bodies in contact with the beam opposing its motion. Care must be taken to assign to them the directions, not of the action of the beam upon the obstacle, but of *the reaction of the obstacle* upon the beam. For the horizontal plane, this reaction is vertically upward as represented by $R$, because the plane is smooth. For the cord, it is the required tension $T$ in the direction of the cord. Finally, the action at $D$ is perpendicular to the beam for the same reason that the action at $A$ is perpendicular to the ground, and the arrow represents the action of the fixed point $D$* upon the beam.

**108.** Since, in the right triangle $ACD$, $AC = 4$ and $CD = 3$, we have $AD = 5$, and denoting the angle $DAC$ by $\alpha$,

$$\sin \alpha = \tfrac{3}{5}, \qquad \cos \alpha = \tfrac{4}{5};$$

$\alpha$ is thus a known angle, and the directions of all the forces are known. There are therefore in this problem only *three* unknown quantities, namely, the magnitudes of the three resistances $R$, $T$ and $S$.

* Although we speak of $D$ as a fixed *point*, there are really two surfaces in contact, just as if $D$ were a round peg; and when the contact is smooth, the direction of the mutual action is a common normal to the surfaces at their point of contact.

We have seen in § IV that equations of equilibrium may be found by resolving forces in various directions. But it is shown in Art. 73 that we can in this way obtain but two *independent* equations; hence it is necessary in this case to obtain *at least one* new condition by the principle of moments. Resolving vertically and horizontally for two conditions, we have,

from vertical forces, $W = R + S \cos \alpha,$

from horizontal forces, $T = S \sin \alpha.$

For the third condition, it is convenient to take the origin of moments at $A$ where $R$ and $T$ have no moments; thus,

from moments at $A$, $W \times AM \cos \alpha = S \times AD.$

Substituting the numerical values of $AM$, $AD$ and the trigonometrical functions, we have

$$W = R + \tfrac{4}{5}S, \qquad (1)$$

$$T = \tfrac{3}{5}S, \qquad (2)$$

$$\tfrac{12}{5}W = 5S, \qquad (3)$$

from which by elimination we obtain

$$S = \tfrac{12}{25}W, \qquad T = \tfrac{36}{125}W, \qquad R = \tfrac{77}{125}W.$$

## Number of Independent Conditions.

**109.** The solution given above shows that, in a problem involving the equilibrium of a solid body under the action of coplanar forces, *three unknown quantities* may be determined by means of three equations of condition, of which one must be derived from the principle of moments.

It may furthermore be shown that, if three conditions thus found are satisfied, all other equations found by taking moments *must* be satisfied. For, using the notation of Art. 100, when the

two equations derived by resolving are satisfied, $R$ (the resultant of the system considered merely as a vector) vanishes, so that the system is either in equilibrium or else equivalent to a couple. But when, by the third condition, the moment about any one point vanishes, the resultant is not a couple; therefore the system is in equilibrium, and the moment about *every* point is zero.

It follows that a problem of this kind is indeterminate if more than three independent quantities are unknown.

110. It is to be noticed that, although the equations derived by resolution are generally the most simple, yet two or even all three of the independent conditions may be found by taking moments. For, let us consider what follows when it is known that the moment of a given system of forces about a given point $A$ vanishes. The resultant of the system, in this case, cannot be a couple, but may be a force whose line of action passes through $A$. If now the moment of the system about some other point $B$ is also known to vanish, the line of action of the resultant, if there be one, must be the line $AB$. When these two conditions are given, the sum of the resolved forces in a direction perpendicular to $AB$ necessarily vanishes, and so does the resultant moment about any point in the line $AB$. But an independent third condition is furnished by the vanishing of resolved forces in any other direction, or of moments about any point not collinear with $A$ and $B$.*

## Choice of Conditions.

111. When only one or two of the three unknown quantities are required, it is desirable to use conditions which are inde-

* The vanishing of the resolved force in a given direction is but the limiting form of the vanishing of the moment about a distant point. Thus, the vanishing of horizontal forces may be said to express the vanishing of the moment about an infinitely distant point in the vertical direction. Hence, the conditions are always the vanishing of the moments about three points, and the conditions are independent when the three points do not lie in a straight line.

pendent of one or more of the unknown quantities whose value is not required, because we shall then be able to employ a smaller number of equations. In the case of a condition obtained by resolving forces, we have seen in Art. 72 that this is done by resolving perpendicularly to the line of action of the force which is to be avoided. In the case of a condition obtained by taking moments, it is done by choosing a point on the line of action for the origin of moments. Thus, in the example solved in Art. 108, supposing $T$ only to be required, we may avoid introducing $R$ and write only equations (2) and (3). Again, if $S$ only were required, equation (3) would suffice, since by taking moments about $A$ we have avoided both $R$ and $T$.

### Case of Three Forces.

**112.** When the number of forces acting in a plane upon a solid body in equilibrium is but three, the lines of action must either meet in a point or be parallel; for, if two of the lines intersect, the forces in these lines have no moment about the point of intersection; hence the third force can have no moment about the same point. Thus the principle that: *The lines of action of three forces in equilibrium must meet in a point* is equivalent to a condition of equilibrium derived from the principle of moments.

**113.** This form of the condition is of frequent application when one of the unknown quantities determines the direction of one of the forces acting at a given point.

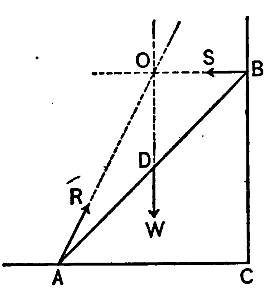

FIG. 40.

For example, let the uniform heavy rod $AB$, Fig. 40, rest in a vertical plane with its upper end $B$ against a smooth vertical wall to which it is inclined at an angle of 45°, and the lower end held at a fixed point $A$ in such a manner that there is *no resistance to turning exerted at the point.* This is frequently expressed by supposing the end of the rod fastened

to $A$ by a *smooth* hinge. Thus the action at $A$ is simply a force acting on the rod at that point in a direction as yet unknown. The only other forces are the weight acting at the middle point $D$, and the action of the wall, which is horizontal because the wall is smooth. Producing the known lines of action to meet in $O$, $AO$ is the line of action of the resistance at $A$. The geometry of the figure now shows that the inclination to the horizontal of the force acting at $A$ is $\tan^{-1} 2$. Thus one of the three unknown quantities has been determined by a single condition; the other two may now if required be found by the other conditions of equilibrium, or by a triangle of forces.

It is obvious that in this problem we might have employed as in the problem of Art. 107, the horizontal and vertical components of the force at $A$, for two of the unknown quantities, instead of its direction and magnitude.

**114.** The principle is particularly useful in problems where one of the unknown quantities determines *the position of equilibrium* of a movable rigid body under given circumstances.

For example, suppose the uniform heavy rod $AB$, Fig. 41, of length $2a$, to be in equilibrium in a vertical plane, with its lower end $A$ against a smooth vertical wall (perpendicular to the plane of the diagram), and resting at some point of its length upon a smooth horizontal rail (piercing the plane of the diagram at $D$) parallel to the wall and at a distance $b$ from it. It is required to determine the inclination $\theta$ of the rod to the horizon.

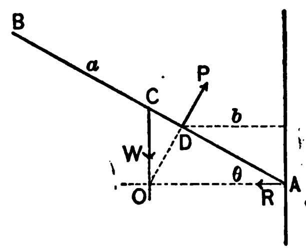


FIG. 41.

The forces acting on the rod are its weight $W$, acting vertically at the middle point $C$, the resistance $R$ of the wall, acting horizontally at $A$ because the wall is smooth, and the resistance $P$ of the rail, acting at right angles to the rod because the rail is smooth. Hence, to be in equilibrium, the position of the rail must be such that the lines of action of these three

forces meet in a point $O$ as represented. We have therefore $OA = a \cos\theta$, $AD = a \cos^2\theta$, and hence

$$b = a \cos^3\theta,$$

which determines $\theta$. The values of $P$ and $R$ may now be found by resolving vertically and along the rod, namely:

$$P = W \sec\theta, \qquad R = W \tan\theta.$$

## Equilibrium of Parallel Forces.

**115.** When all the forces are parallel and in a single plane, the resolved forces in that direction in the plane which is at right angles to the lines of action vanish. In this case, then, we have to consider but two conditions of equilibrium, one at least of which must be obtained by taking moments. The most familiar examples are those in which the forces are the weights of bodies applied at different points of a rigid body, such as a beam in a horizontal position, the two unknown quantities being the magnitudes of the upward supporting forces, or the position and magnitude of a single force.

Thus, let the beam $AB$, weighing 280 pounds and 20 feet long, be supported at its ends, and loaded with a weight of 200 pounds at a point 4 feet from $A$, a weight of 320 pounds at a point 5 feet from $B$, in addition to its own weight acting at the middle point; required the reactions, $P$ and $Q$, of the supports at $A$ and $B$. Taking moments about $A$, we obtain a condition of equilibrium independent of $P$, namely,

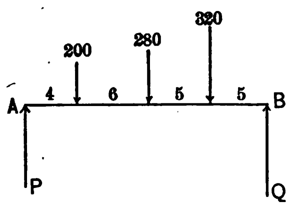


FIG. 42.

$$20Q = 4 \times 200 + 10 \times 280 + 15 \times 320 = 8400,$$

whence $Q = 420$ pounds. $P$ may be found in like manner, or

more simply (having found $Q$) from the equation of vertical forces,

$$P + Q = 800,$$

whence $P = 380$ pounds.

**116.** We may also proceed as follows, which is sometimes more convenient. Resolving each of the downward forces into components acting at $A$ and $B$, $P$ is evidently the sum of the components acting at $A$, and $Q$ is the sum of those acting at $B$. Thus, the 200 pounds is divided into components inversely proportional to 4 and 16, the distances of its point of application from $A$ and $B$; that is to say, $\frac{4}{5}$ of it, or 160 pounds, is the component at $A$, and $\frac{1}{5}$ of it, or 40 pounds, the component at $B$. In like manner, $\frac{1}{2}$ of the 280 pounds, or 140 pounds, is the component of this force at $A$, and 140 pounds acts at $B$. Finally, $\frac{1}{4}$ of the 320 pounds, or 80 pounds, acts at $A$, and $\frac{3}{4}$ of it, or 240 pounds, acts at $B$. Hence, adding the like components, we have

$$P = 160 + 140 + 80 = 380, \qquad Q = 40 + 140 + 240 = 420.$$

EXAMPLES. VI.

1. A uniform beam of weight $W$ rests against a smooth vertical wall, and a smooth horizontal plane with which it makes the angle $\alpha$. Its lower end is attached by a string to the foot of the wall. Find the tension of the string. $\frac{1}{2}W \cot \alpha$.

2. The ends $A$ and $B$ of a uniform rod, of length $2b$ and weight $W$, are fastened by strings, whose lengths are $2a$ and $a$, respectively, to the point $C$ in a smooth vertical wall, the rod and strings lying in a vertical plane perpendicular to the wall, against which $B$ rests below $C$, as in Fig. 43. Denoting by $\phi$ the inclination of the longer string to the wall, find the tensions $S$ and $T$ of the strings.

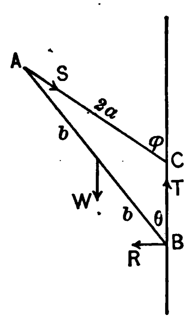

FIG. 43.

$$S = W; \;\; T = W(1 + \cos \phi) = W\left(\frac{b^2}{a^2} - \frac{1}{4}\right).$$

3. A uniform beam of weight $W$, and length 3 feet, rests in equilibrium with its upper end $A$ against a smooth vertical wall, while its lower end $B$ is supported by a string, 5 feet long, whose other end is attached to a point $C$ in the wall. Find $AC$ and the tension of the string.

$$AC = \frac{4}{\sqrt{3}} \text{ feet}; \quad T = \frac{5\sqrt{3}}{8} W.$$

4. A uniform lower boom $AB = l$, whose weight is $W$, is supported at $C$ in a horizontal position by a topping-lift, inclined at 45°. Given $AC = \frac{3}{4}l$, find the tension of the lift, and the thrust at the goose-neck $A$.

$$\frac{2\sqrt{2}}{3} W; \quad \frac{\sqrt{5}}{3} W.$$

5. Prove that three forces acting at the middle points of the sides of a triangle perpendicularly inward, and proportional to the sides, are in equilibrium, and extend the theorem to a plane polygon of any number of sides.

6. A uniform beam of length $l$ rests upon the horizontal rim of a hemispherical bowl of radius $a$, with its lower end upon the smooth concave surface. Determine its inclination $\theta$ to the horizon when in equilibrium.

$$\cos\theta = \frac{l}{16a} + \frac{\sqrt{(l^2 + 128a^2)}}{16a}.$$

7. A bar, whose length is $l$, rests upon a smooth peg at the focus of a parabola whose axis is vertical and whose parameter is $4a$, with its lower end on the smooth concave arc. Find its inclination to the axis.

$$\cos^4 \tfrac{1}{2}\theta = \frac{a}{l}.$$

8. A uniform rod $AB$, of length $4a$, has the end $A$ in contact with a smooth vertical wall, and one end of a string is fastened to the rod at the point $C$, such that $AC = a$, while the other end is fastened to the wall. Show that, in order that equilibrium may exist, the string must have a definite length, but that its inclination is indeterminate.

9. A uniform yard-stick weighing 10 ounces is supported in a horizontal position by the thumb at one end, and the forefinger at a point 3 inches from the end. What is the pressure on the thumb and on the finger?

50 oz.; 60 oz.

10. If, in Fig. 39, the cord is attached to the post at a point 1 foot from the ground, find the values of $R$ and $T$, supposing $W = 250$ pounds. $T = 18\sqrt{17}$ lbs.; $R = 136$ lbs.

11. A uniform beam $AB$, 3 feet long, weighing 10 pounds, rests in equilibrium with 4 pounds hanging from $A$ and 13 pounds from $B$. What is the position of the point of support?
2 feet from $A$.

12. The scales $A$ and $B$ of a false balance are at unequal distances, $a$ and $b$, from the fulcrum or point of support, but they balance when empty. Find the true weight $W$ of a body whose weight appears to be $P$ when placed in the scale $A$, and $Q$ when placed in the scale $B$; find also the ratio of $b$ to $a$.

$$W = \sqrt{(PQ)}; \quad \frac{b}{a} = \sqrt{\left(\frac{Q}{P}\right)}.$$

13. If, in example 12, the scales do not balance when empty, $A$ tending downward with the moment $H$, find the ratio $b$ to $a$ and the value of $H$, supposing the true weight $W$ known.

$$\frac{b}{a} = \frac{W - Q}{P - W}; \quad H = a\frac{W^2 - PQ}{P - W}.$$

14. A beam $AB$ weighing $\frac{1}{2}$ ton per running foot and 18 feet long is loaded with 4 tons at $A$ and 5 tons at $B$; it is supported at points 4 feet from $A$ and 6 feet from $B$. Find the supporting forces $P$ and $Q$. $P = 5\frac{5}{8}$ tons; $Q = 12\frac{3}{8}$ tons.

15. A topgallant yard 40 feet long, weighing 1680 pounds, is supported in a horizontal position by lifts attached to the ends, which if produced would meet in a point 21 feet above the yard. Find the tension on either lift. 1160 pounds.

16. A uniform plank 20 feet long, weighing 42 pounds, is placed over a rail; two boys weighing 75 and 99 pounds, respectively, stand each at a distance of a foot from one end. Find the position of the rail for equilibrium.
1 foot from the middle point.

17. A uniform rod 22 feet long, weighing 80 pounds, rests with its upper end against a smooth vertical wall and its lower end supported by a cord, 26 feet long, attached to a point in the wall. Find the tension of the cord. 130 pounds.

18. An iron rod weighing 4 lbs. per linear foot projects from a cask 3 feet in diameter and 4 feet high, the part within the cask forming a diagonal of the vertical section through the axis. The weight of the cask is 60 pounds and is assumed to act at its centre. Find the length of the rod if the cask is on the point of overturning. 15 feet.

## VII.

## The Rigid Body regarded as a System of Particles.

**117.** When a limited number of forces act upon a body at rest, each of the several points of application may be regarded as a particle in equilibrium. In this point of view, the forces which act upon the body (including of course the resistances of other bodies) are called the *external forces*. The forces which act at any one of the points of application, and there produce equilibrium consist of some of the external forces together with the resistances of other parts of the body. These resistances are called *the internal forces*.

The whole set of internal forces consists of actions and reactions between the several parts of the body, forming *stresses*; of these the simplest kind are the *tensions* and *compressions* considered in Arts. 26 and 27. The nature of the stresses thus produced in a body by external forces depends partly on the material and form of the body; and their study, together with that of the changes of form the body may undergo before reaching a state of equilibrium, forms a special branch of the Science of Mechanics, namely, *the Mechanics of Structures and Materials.*

**118.** But when we are, as at present, considering only the equilibrium of the external forces, we may imagine the simplest possible rigid connection to exist between the several points of application. For example, in Fig. 24, p. 65, the forces $P_1$, $P_2$, $P_3$ and the force $-R$ (which is equal and opposite to the resultant $R$, and in the same line of action) form a system of four forces in equilibrium. Now, if we take $A$ as the point of application for

$P_1$ and $P_2$, and $B$ as the point of application of $P_3$ and $-R$, a single rod joining $A$ and $B$ will be sufficient to form the rigid connection, and this is the simplest body upon which the four external forces could act so as to produce the equilibrium. This rod will be in a state of compression, and the two phases of this stress, which are equal and opposite by the law of reaction, namely, $Q$ acting at $B$ and $-Q$ acting at $A$, are the internal forces. Altogether then we have six forces, forming two sets of three each, which are in equilibrium at $A$ and at $B$ respectively.

**119.** An equally simple mode of practically illustrating the equilibrium of the four external forces in Fig. 24 would be to suppose $P_2$ and $-R$ to act at the intersection of their lines of action, and $P_1$ and $P_3$ to act at the intersection of their lines of action. Then if we connect these points by a rod, it will be in a state of tension, and there will as before be three forces in equilibrium at each end.

But if the four external forces acted at any other points of their respective lines of action, taken for instance upon a thin plate in the plane of the forces, the stresses produced in the plate, would be of a much more complicated nature.

**120.** Conversely, a system of particles having a mutual action, as, for example, the weights $P$ and $Q$ in the example of Art. 81 p. 59, may be replaced by a solid body. Thus, in Fig. 23, we may consider the rod $AB$ as a solid in equilibrium, acted upon by the four external forces $P$, $Q$, $R$ and $S$, excluding the two forces equal to $T$, which are now regarded as internal forces. The problem thus becomes one of three unknown quantities, $R$, $S$ and $\theta$, to be determined by the three conditions of equilibrium for forces in a plane (acting on a solid); whereas it was before treated as one of four unknown quantities, $R$, $S$, $\theta$ and $T$, to be determined by four conditions, namely, two for the equilibrium of each particle.

When a single unknown quantity, for example, $\theta$ in this problem, is required, we can frequently use a smaller number of equations. Thus, in the solution given in Art. 81 we obtained two equations independent of $R$ and $S$, and so had only to

eliminate $T$. On the other hand, treating the bar as a solid, we can obtain two equations independent of $R$ by resolving along $CA$ and taking moments about $A$. Denoting the length of the rod by $a$, these equations are

$$S = (P + Q) \sin \alpha,$$
$$Sa \sin \theta = Qa \cos (\alpha - \theta);$$

and these will be found, on eliminating $S$, to give the result already found in Art. 81.

## The Funicular Polygon for Parallel Forces.

**121.** We have seen in Art. 90 that the graphic construction employed in Fig. 28, p. 69, to find the resultant of a number of parallel forces in a plane (supposed in the figure to be the weights of given bodies acting in given lines), gives rise to a broken line or polygon, which is called *funicular* because it is the form of a cord to which the weights may be knotted and sustained in equilibrium by means of supporting forces at the ends of the cord. Regarding the cord as a rigid body, the external forces in Fig. 28 are the weights and the oblique forces, $Q$ and $Q'$ which are applied to the ends of the cord by means of two fixed points to which they are attached.

**122.** In Fig. 44 we give a modification in which the funicular polygon is *closed* by a bar connecting the ends of the cord. This permits the supporting forces to be vertical, and to act, like the weights, in given lines. The data taken are the same as those of the problem solved in Art. 115. (See Fig. 42.) The lines of action, both of the weights and the upward forces, are drawn at their proper distances in the right-hand figure. The left-hand figure, or *force diagram*, is constructed by laying off $AB$, $BC$, $CD$ in a vertical line, to represent on a selected linear scale the weights taken in order; and then joining $A$, $B$, $C$ and $D$ to a point $O$, called *the pole*, taken at random. Then, starting from any point $M$ in the line of action of $P$, the funicular polygon

$M123N$ is constructed as in Art. 89; that is to say, its sides are drawn successively parallel to $AO$, $BO$, $CO$ and $DO$. Finally, the polygon is closed by the line $MN$, and $OX$ is drawn parallel to $MN$ in the force diagram.

**123.** Suppose now the closed polygon to be composed of bars connected by smooth joints or hinges, and let us determine conversely what external forces applied at the joints are necessary in order that the several links of the chain from $M$ to $N$ may be subjected to tensions proportional to the lines to which they are parallel in the left-hand diagram.

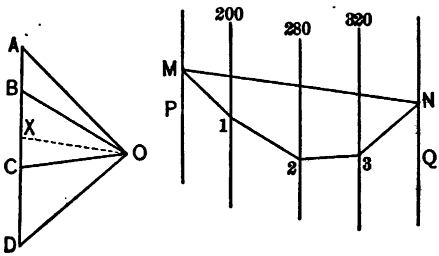


FIG. 44.

Since two sides of the triangle $ABO$ are parallel and proportional to the two internal forces we have supposed to act at the joint 1, it is a triangle of forces for the equilibrium of that joint; therefore $AB$ represents the required force there acting. In like manner, $BCO$ is a triangle of forces for the equilibrium of the joint 2; the forces acting in the directions $AB$, $BO$, $OA$ in the first case, and in the directions $BC$, $CO$, $OB$ in the second. Moreover, since the line $BO$ represents in these two triangles the action of the joint 2 upon 1, and that of 1 upon 2, *which are equal by the law of reaction*, these triangles represent the several forces *upon the same scale*. It follows that the forces at the joints 1 and 2 necessary to complete the equilibrium must act vertically downward and be represented on the same scale by $AB$ and $BC$.

In like manner, at all the joints the necessary external forces must be weights proportional to the segments of the vertical line. Now, in constructing the figure these segments were drawn to represent the given weights; hence the polygon is capable of supporting the given weights in their given lines of action.

124. Moreover, if we suppose $MN$ subject to a compression represented on the same scale by $OX$, the triangles $AOX$, $XOD$ will be triangles of forces for the equilibrium of the points $M$ and $N$. Hence vertical supporting forces represented on the same scale by $XA$ and $DX$, acting at $M$ and $N$, will complete the equilibrium of the whole polygon. Thus the process is a graphic method of determining the supporting forces $P$ and $Q$, which were (for the data of this problem) found in Art. 116, by the method of moments, to be 380 and 420 pounds respectively.

If the lines from the pole $O$ had been drawn to represent any other stresses, the sides of the closed polygon $ABCDX$ would still have represented vectorially the external forces necessary to produce these stresses.

### The Funicular Polygon for Forces not Parallel.

125. In the preceding articles the system of external forces, which is itself in equilibrium, is a system of parallel forces, and we have seen that it is possible to draw a funicular polygon (having its vertices upon the given lines of action), which may be regarded as a chain of jointed bars in equilibrium under the action of the given forces acting at the joints. In precisely the same way a funicular polygon may be drawn for any coplanar system of forces in equilibrium. For this purpose, the given forces are taken in a certain order, as $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ in Fig. 45, and the polygon of forces $ABCDE$ is drawn by laying off the corresponding vectors in the selected order from any point $A$. This will be a closed polygon if the forces are in equilibrium. A pole $O$ is then taken at random in the plane of the vectorial or force polygon, and joined to the several vertices. Then, starting from any point in the line of action of $P_1$ a parallel to $OB$ is drawn to intersect the line of action of $P_2$. This is a side of the funicular polygon. In like manner the other sides are drawn successively, each corresponding to a vertex of the force polygon. Then, if the forces are in equilibrium, the parallel last drawn will

pass through the point on the line of action of $P_1$ from which we started; that is, the funicular polygon will *close* as represented in the diagram. The proof is precisely the same as in Art. 123.

The condition that the vectorial polygon shall close is the graphical equivalent of the condition $R = 0$ of Art. 100, which is equivalent to two analytical conditions. Hence the condition that the funicular polygon shall close is the graphical equivalent of the third analytical condition $K = 0$,* which was derived from the principle of moments.

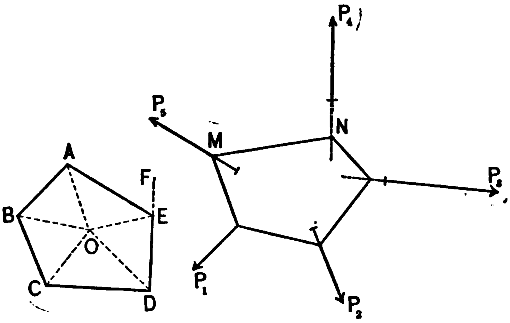

FIG. 45.

**126.** The funicular polygon is practically employed in determining three unknown elements in a system of forces in equilibrium. For example, if only the point of application $M$ of $P_5$ were known (its direction and magnitude being two unknown elements), and only the line of action of $P_4$ (its magnitude being a third unknown quantity), we could complete the two diagrams as follows: Starting from $A$, the data enable us to draw $ABCD$, and through $D$ the line $DF$ parallel to $P_4$ upon which $E$ must lie. Then, selecting the pole, we join $AO$, $BO$, $CO$ and $DO$. We can now, in the funicular polygon, starting from $M$, draw all the sides but one in the usual manner, arriving at a point $N$ on the given line of action of $P_4$. Joining $MN$, we have the final side of the funicular polygon. Finally, returning to the vectorial

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* The closing of the vectorial polygon is equivalent to two conditions because the final side must have a given direction and a given length; but the closing of the funicular polygon is but a single condition, because the final side is only required to have a certain direction.

diagram; we draw $OE$ parallel to this final side; thus determining the point $E$ and completing the force diagram. The length of $DE$ and the length and direction of $EA$ give, respectively, the magnitude of $P_4$, and the magnitude and direction of $P_6$ which were to be determined.

## The Suspension Bridge.

**127.** In the suspension bridge the weight of a uniform platform is carried by vertical rods to a cable consisting of jointed bars. Supposing the vertical rods to be spaced at equal horizontal distances and attached to the joints, the cable becomes, when the weight of the bars is neglected, a funicular polygon for the case of equal parallel forces acting in equidistant lines, as represented in Fig. 46. The extremities $M$ and $N$ are fixed

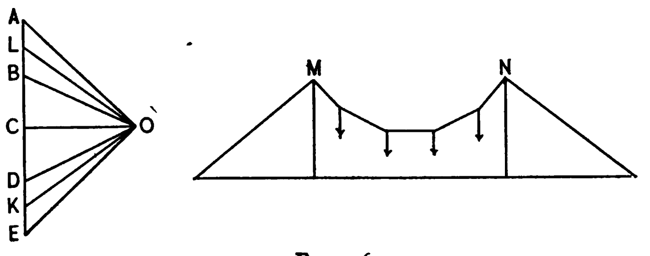


FIG. 46.

at the same level at the tops of solid piers symmetrically situated with respect to the weights. If the slopes of the extreme bars are given, the pole $O$ is found by the intersection of $AO$ and $BO$ parallel to the bars; then $OB$, $OC$, etc., give the direction and stress in the intermediate bars. The perpendicular $OC$ will represent the stress of the middle bar (if there be an odd number of bars), which will be horizontal. This stress, which we shall denote by $H$, is the horizontal component of the stress in every bar. Instead of a closing line $MN$, of which the compression would be $H$, the horizontal component of the pull of the end

links at $M$ and $N$ is balanced by the like components of the tensions of ties connecting $M$ and $N$ with fixed points.

Now, drawing $OK$ and $OL$ in the force diagram parallel to these ties, and supposing the piers to take only vertical compression, the equilibrium of the point $M$ is represented by the triangle $OKA$, and that of $N$ by the triangle $OEL$. Thus $OK$ and $LO$ give the tension of the ties, and $KA$, $EL$ give the supporting forces supplied by the piers. Each of these exceeds the half sum of the weights suspended from the cable by the vertical component of the tension of the tie.

### Form of the Suspension Cable.

**128.** If the chain be replaced by a perfectly flexible cord, and the number of points at which the weight is applied be increased indefinitely, the limiting form to which the cord approaches will be a continuous curve.

To find this limiting form, let the lowest point $O$, Fig. 47, be taken as the origin of rectangular coordinates, and the horizontal tangent at that point as the axis of $x$. The portion of the cord $OP$ between the origin and any point $(x, y)$ of the curve would remain in equilibrium if it became rigid. Now, considered as a rigid body, it is acted upon by three

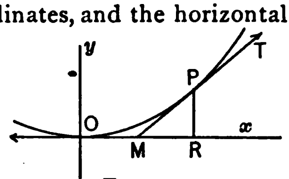


FIG. 47.

external forces, namely, the resultant of the entire weight suspended from it, and the tensions at $O$ and $P$, which are the actions of the other parts of the cord upon it. The weight suspended from each unit of length in the horizontal projection of the cable is by hypothesis constant. Denoting it by $w$, the whole weight suspended from $OP$ is $w \,.\, OR$, or $wx$; and, because it is uniformly distributed, the line of action of the resultant bisects $OR$ in $M$. Since the lines of action of the three forces must meet in one point, it follows that the line of action of $T$, the tension at $P$, must pass through $M$, the intersection of the

lines of action of the weight $wx$ and the horizontal tension $H$ at $O$. Then, because the ordinate $PR$ is vertical, the triangle $PMR$ is a triangle of forces; and since $MR = \frac{1}{2}x$, we have

$$\frac{H}{wx} = \frac{\frac{1}{2}x}{y}.$$

Hence, the equation

$$Hy = \tfrac{1}{2}wx^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is true for every position of the point $P$, is the required equation of the curve. The curve is therefore a parabola with its vertex at $O$ and its axis vertical.

**129.** The constant $H$ in this equation is readily determined if the coordinates of any one point as referred to $O$ are known. Thus, if $l$ denotes the total length $MN$, or *span*, and $h$ denotes the depth of the parabolic arc (or distance of $O$ below the line $MN$), the coordinates of the extremity of the cable, corresponding to $N$ of Fig. 46, are $\frac{1}{2}l$ and $h$. Substituting in equation (1) these values of $x$ and $y$, we find

$$H = \frac{wl^2}{8h} = \frac{Wl}{8h}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where, in the third member, $W$ is the whole weight suspended upon the cable.

Denoting by $\alpha$ the inclination to the horizontal of the cable at this point, we have, from Fig. 47,

$$\tan\alpha = \frac{y}{\frac{1}{2}x} = \frac{4h}{l};$$

and the tension at this point is

$$T_1 = H\sec\alpha = W\frac{\sqrt{(l^2 + 16h^2)}}{8h},$$

which is the greatest tension of the cable.

For example, if the span is 100 feet and the depth 10 feet, we find $H = \frac{5}{4}W$ and $T_1 = \frac{\sqrt{29}}{4}W$, for the least and the greatest tension of the cable.

**130.** It is obvious that the cable will still be in equilibrium if we regard the portion of it between any two points as a solid connected with the portion on either side by smooth joints. Thus the cable, which is flexible at every point, may be replaced by a chain consisting of jointed bars *provided the joints are situated on the parabolic arc.* The weight is here still supposed to be distributed uniformly along the horizontal projection and to be attached at all points of the bars, exactly as if they were uniform heavy bars (but of weights proportional to their horizontal projections, not to their lengths). Under these circumstances there will be at each joint only two forces acting, namely, the equal action and reaction of the two bars, which will be in the direction of the tangent to the parabola.

**131.** Suppose now that the weight suspended from any one bar be divided into two equal parts and applied directly to the pins which constitute the joints at its two ends. The resultant of the vertical forces will not be changed, and therefore the equilibrium will not be disturbed. When all the weight is thus concentrated at the joints we have the funicular polygon of Fig. 46, which is therefore a polygon inscribed in a parabola. When the number of bars is considerable the depth of the polygon will not differ sensibly from that of the parabolic arc, and the tensions (which are now in the directions of the bars themselves) do not differ sensibly from the mutual actions mentioned in the preceding article. Therefore the formulæ of Art. 129 may be used in this case also.

## The Catenary.

**132.** The form assumed by a uniform heavy and perfectly flexible cord hanging from two fixed points is called a *catenary.* We refer the curve to rectangular co-ordinates as in Art. 128, the vertex, or lowest point, $O'$ being the origin, Fig. 48; and, as before, regard the portion $OP$ as acted upon by three forces. These are the horizontal tension $H$ at $O'$, the tension $T$ at $P$ in the direction of the tangent to the curve, and the resultant of the forces of gravity acting upon the portion $O'P$. This last force must as before act in a vertical line passing through the intersection $M$ of the tangents. Denoting by $w$ the weight of the cord per unit length, and by $s$ the length of the arc measured from $O'$, the weight of the portion $O'P$ is $ws$. The horizontal component of $T$ is equal to the constant $H$, and its vertical component is $ws$; therefore

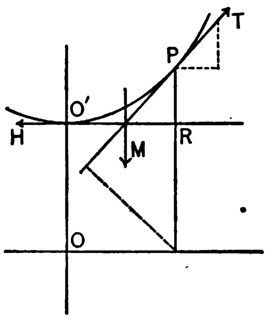

FIG. 48.

$$T = \sqrt{(w^2s^2 + H^2)}.$$

**133.** The triangle $PMR$ is, as in Fig. 47, a triangle of forces; but, in this case, we do not know the position of $M$. Employing, however, the differential triangle whose sides $dx$, $dy'$, and $ds$ are parallel to the three forces, we have

$$dx : dy' : ds = H : ws : \sqrt{(w^2s^2 + H^2)}. \quad . \quad . \quad . \quad (1)$$

The differential relation between $y$ and $s$ is here the most simple, giving

$$dy' = \frac{wsds}{\sqrt{(w^2s^2 + H^2)}};$$

whence, by integration,

$$wy' = \sqrt{(w^2s^2 + H^2)} + C.$$

To determining the constant of integration, we notice that $s = 0$ when $y' = 0$; whence $C = -H$, and

$$wy' = \sqrt{(w^2s^2 + H^2)} - H. \quad . \quad . \quad . \quad . \quad (2)$$

It is convenient to introduce, in place of $H$, the constant $c$ such that

$$wc = H, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and then to take a new origin $O$ at the distance $c$ below $O'$. Equation (2) then becomes

$$w(y' + c) = wy = \sqrt{(w^2s^2 + w^2c^2)}, \quad . \quad . \quad . \quad . \quad (4)$$

or

$$y^2 = s^2 + c^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Equation (3) shows that $c$ is the length of that portion of the cord whose weight is $H$, the tension at the vertex $O'$ of the curve; and, since the third member of equation (4) is the value of $T$ (Art. 132), $y$ is equal to that length of cord whose weight is $T$. Thus the tension at any point of the curve is equal to the weight of a portion of the cord whose length is equal to the ordinate as referred to the new origin. Hence, if there were two smooth pegs at any two points of the curve, a portion of the cord equal in length to the arc and the two ordinates would hang in equilibrium over the pegs with its two extremities on the new axis of $x$.*

**134.** To obtain the equation of the curve as referred to the origin $O$, we eliminate $s$ by equation (5) from the relation

* It is interesting to notice that, if the portions whose weights produce the tensions did not end at the same level, we should, by connecting the extremities by another portion of the same cord passing around smooth pegs at the level of the lower extremity, have a perpetual motion, because this new portion of cord would not be in equilibrium.

between $dx$ and $dy'$ (or $dy$, since $d(y-c)=dy$) in equation (1), Art. 133. The result is

$$\frac{dy}{dx}=\frac{ws}{H}=\frac{\sqrt{(y^2-c^2)}}{c},$$

or

$$\frac{dy}{\sqrt{(y^2-c^2)}}=\frac{dx}{c}.$$

Integrating, and determining the constant so that $y=c$ when $x=0$,

$$\log\frac{y+\sqrt{(y^2-c^2)}}{c}=\frac{x}{c},$$

or

$$\frac{y+\sqrt{(y^2-c^2)}}{c}=e^{\frac{x}{c}}.$$

Taking the reciprocal of this equation,

$$\frac{y-\sqrt{(y^2-c^2)}}{c}=e^{-\frac{x}{c}},$$

whence, by addition,

$$y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

is the equation of the catenary.

Since, by equation (5), $s=\sqrt{(y^2-c^2)}$, the equations above give

$$s=\frac{c}{2}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

a result which can also be obtained directly by the integration of the differential relation between $x$ and $s$ in equation (1), Art. 133.

### Approximate Formulæ.

**135.** When the two fixed points to which the cord is attached are at the same level, let their distance apart, or the *span* of the

catenary, be denoted by $l$, and the distance of the vertex below the horizontal line of supports by $h$; then, $x$ and $y$ referring to the right-hand support, $l = 2x$ and $h = y - c$. Since

$$e^z = 1 + z + \frac{z^2}{2\,!} + \frac{z^3}{3\,!} + \dots,$$

we have, by expansion, from equation (6),

$$h = y - c = c\left(\frac{x^2}{2c^2} + \frac{x^4}{24\,c^4} + \dots\right).$$

When $h$ is very small compared with $x$, $c$ is very large, and the first term gives a close approximation, namely,

$$h = \frac{x^2}{2c}; \qquad \text{whence} \qquad c = \frac{x^2}{2h} \quad . \quad . \quad . \quad . \quad (1)$$

may be taken as the value of $c$ for a *flat* catenary (that is, one which is nearly a straight line) whose length is $2x$ and whose depth is $h$. Since $H = wc$, this gives for the tension of the cord, which in this case does not differ sensibly throughout the curve,

$$H = \frac{wx^2}{2h} = \frac{wl^2}{8h} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This agrees with the formula of Art. 129 for the suspension bridge, as we should expect, since the weight is now distributed nearly uniformly along the axis of $x$.

**136.** An approximate formula for $c$, when $s$ and $x$ are given and their difference is very small, is obtained by expanding equation (7) of Art. 134; thus,

$$s = c\left(\frac{x}{c} + \frac{x^3}{3\,!\,c^3} + \dots\right) = x + \frac{x^3}{6c^2} + \dots,$$

whence, taking two terms of the expansion,

$$c = \sqrt{\frac{x^3}{6(s - x)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, substituting this in the approximate expression for $h$, we obtain

$$h = \tfrac{1}{2}\sqrt{[6x(s - x)]} = \tfrac{1}{4}\sqrt{[6l(S - l)]}, \quad . \quad . \quad . \quad (4)$$

where in the third member $S$ denotes the whole length, and $l$ the span.

For example, let the length of a wire suspended between two points 100 feet apart in a horizontal line be 100 feet 1 inch; to find the depth $h$ of the middle point below this line. Here $l = 100$, $S - l = \frac{1}{12}$, and, substituting in equation (4), $h = \frac{5}{4}\sqrt{2}$, or about 1 foot 9.2 inches. The tension of this wire may now be found from equation (1), or directly from equation (3) because $H = cw$. It is $H = 500\sqrt{2}w$, that is, the weight of about 707 feet of the wire.

**137.** When $h$ and $x$ are given and $h$ is small, the difference $s - x$ is very small, and equations (3) and (1) give the very close approximation

$$s - x = \frac{2h^2}{3x}; \quad \text{whence} \quad S - l = \frac{8h^2}{3l}. \quad . \quad . \quad (5)$$

## Equilibrium of a System of Solid Bodies.

**138.** A number of movable solid bodies, between which such mutual actions exist that they can only occupy certain positions relatively to one another and to fixed bodies, form a *system of solids*; and the possible positions are called the *configurations* of the system. When some or all of the bodies are acted upon by forces *external to the system*, it may happen that equilibrium can exist only when the system is in a particular configuration. The mutual actions which exist between the bodies of the system are the *internal forces*. The external forces of the system will them-

selves be in equilibrium, exactly as if the system formed a single body. But it is only by considering the separate bodies that we can determine the configuration of equilibrium.

**139.** For example, suppose four uniform bars, each of length $2a$ and weight $W$, jointed together so as to form a rhombus, to be suspended as in Fig. 49 from two smooth pegs fixed at the same height and at the distance $2c$ in a vertical wall; and let it be required to determine a position of equilibrium in which the diagonal $AC$ is vertical. The angle $\theta$ between the side $AB$ and the vertical will serve to determine the required configuration. Since the forces act symmetrically upon the pair of upper bars and the pair of lower bars respectively, it is evident that the mutual action of the bars at $A$ and at $C$ is horizontal, and that it is only necessary to consider the equilibrium of the bars $AB$ and $BC$. These bars are drawn separately in the diagram to show more clearly the forces acting upon them. In the case of $AB$, these are four in number; namely, its weight $W$ acting at its middle point, the horizontal action $S$ at $A$, which we shall assume

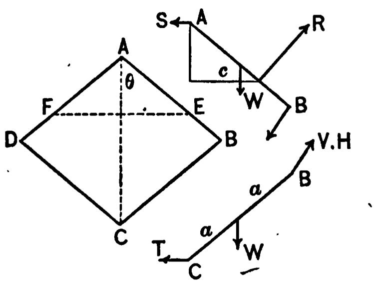


FIG. 49.

to act to the left, the resistance $R$ of the peg at $E$, which acts perpendicularly to the bar (and therefore in a direction depending upon $\theta$), and the action at $B$ of the lower rod, of which the direction as well as the amount is unknown. We may therefore take $V$ and $H$, the vertical and horizontal components of this force, for two of the unknown quantities. In the case of the

lower bar we have these forces $V$ and $H$ acting in the opposite directions at $B$, the weight $W$ acting at the middle point, and the horizontal force $T$ acting at $C$.

Hence there are in all six unknown quantities, $\theta$, $R$, $S$, $T$, $V$ and $H$, and the conditions of equilibrium, three for each bar, are sufficient in number to determine them.

The equilibrium of the rod $BC$ gives, by taking vertical and horizontal forces and moments about $B$,

$$\begin{aligned} V &= W, \\ H &= T, \\ 2\,Ta \cos\theta &= Wa \sin\theta; \end{aligned}$$

and the equilibrium of $AB$ gives

$$\begin{aligned} R\sin\theta &= V + W \\ R\cos\theta &= S + H, \\ Rc \operatorname{cosec}\theta &= Wa\sin\theta + 2\,Va\sin\theta + 2Ha\cos\theta. \end{aligned}$$

The first five of these six equations determine the five unknown forces in terms of $\theta$, namely:

$$\begin{aligned} V &= W, & H &= T = \tfrac{1}{2}W\tan\theta, \\ R &= 2\,W\operatorname{cosec}\theta, & S &= W(2\cot\theta - \tfrac{1}{2}\tan\theta); \end{aligned}$$

and then the sixth equation gives

$$c = 2a\sin^2\theta,$$

which determines $\theta$.

If the value of $\theta$ is found to exceed $\tan^{-1}2$, the value of $S$ will be negative and its direction will be opposite to that which we assumed it to be in the diagram.

## EXAMPLES. VII.

1. Solve Ex. IV. 10 by assuming a rigid triangle turning about the centre instead of a circular wire and string.

2. Two uniform and equal smooth cylinders rest in contact, and each is in contact with one of two smooth planes inclined at angles $\alpha$ and $\beta$ to the horizon. Determine the inclination $\theta$ to the horizon of the plane passing through their axes.

$$\tan\theta = \tfrac{1}{2}(\cot\alpha - \cot\beta).$$

3. A polygon composed of bars without weight has the form of five sides of a regular dodecagon with the middle side horizontal. Show that three equal weights may be hung one from the middle of this bar and one from each of the two upper joints.

4. A funicular polygon is inscribed in a semicircle with horizontal diameter, the sides taken in order being chords of 60°, 30°, 30° and 60°. Find the ratio of the weights at the joints.

$$[\sqrt{3} - 1 : 2 - \sqrt{3} : \sqrt{3} - 1 \text{ or}]\ 2 : \sqrt{3} - 1 : 2.$$

5. Determine the ratio of the weights when the sides of the funicular polygon in example 4 taken in order are chords of 60°, 60°, 30° and 30°. $\sqrt{3} : 1 : 1 + \sqrt{3}$.

6. Forces acting in lines bisecting the angles of a plane polygon are proportional to the cosines of the half angles, and all act outward. Prove that they form a system in equilibrium, and that the polygon of forces can, in this case, be inscribed in a circle.

7. Show that, if one of the vertices of the polygon of external forces be taken as the pole, two sides of the funicular polygon coincide with the lines of action of two of the forces, the result being the funicular polygon for the resultant of these two forces and the remaining forces. Compare Fig. 25.

8. In the suspension bridge, the horizontal projections of the bars being equal, prove that the tangents of the inclinations of successive bars are in arithmetical progression; also, if the number of bars is even, the heights of the joints taken in order above the middle joint are proportional to the squares of the integers in natural order.

9. Show, by mechanical considerations, that the vertical

through the intersection of tangents at the extremities of *any* arc of the parabola in Fig. 47 bisects the chord of the arc; and thence that the tangent at the point where this vertical cuts the arc is parallel to the chord and bisects the distance between the intersection of the tangents and the middle point of the chord.

10. Derive the equation for the suspension cable by integration.

11. A bridge of 360 feet span is supported by two suspension cables. The weight of the bridge is $\frac{1}{2}$ ton per foot run, and the dip of each cable is $37\frac{1}{2}$ feet. Find the least and the greatest tension of the cable; also, if the stays and the cable are equally inclined to the vertical at the top of a pier, what is the whole pressure on the pier. 108 tons; 117 tons; 180 tons.

12. Derive the value of $H$, Art. 129, directly from the extreme case in which the suspension cable is reduced to two bars.

13. Prove that the perpendicular from the foot of the ordinate in Fig. 48 has the constant value $c$; and thence that the involute of the catenary is *the tractrix* (the curve in which the part of the tangent between the point of contact and the axis of $x$ is constant).

14. Prove the following values of $x$, $y$ and $c$ in terms of $s$ and $h$ in the catenary where $h = y - c$ as in Art. 134:

$$y = \frac{s^2 + h^2}{2h}, \qquad c = \frac{s^2 - h^2}{2h}, \qquad x = \frac{s^2 - h^2}{2h} \log \frac{s + h}{s - h}.$$

The expanded form of the last expression is

$$x = s\left[1 - \frac{2}{1.3}\frac{h^2}{s^2} - \frac{2}{3.5}\frac{h^4}{s^4} - \frac{2}{5.7}\frac{h^6}{s^6} - \dots\right].$$

15. A cord weighing 2 oz. per linear inch hangs over two smooth pegs at the same level; the tension of the cord at its lowest point, which is 6 inches below the pegs, is 20 oz. Find the whole length of the cord. 56.98 inches.

16. The wire for a line of telegraph cannot sustain more than the weight of 4000 feet of its own length. If there are 22 poles to the mile, what is the least sag allowable? 21.6 inches.

17. How much per mile does the actual length of the stretched wire in example 16 exceed the straight line? 9.504 inches.

18. Two rods, $AC$ and $BC$, of uniform weight per linear inch, are jointed together at $C$ and to two fixed points in the same vertical at $A$ and $B$. Show that the direction of the action at $C$ bisects the angle $ACB$.

19. Two uniform heavy rods have their ends connected by weightless strings and are supported at the middle point of one of them. Prove that in equilibrium either the rods or the strings are parallel.

20. Two equal rods without weight are hinged together at their common middle point, $C$, and placed in a vertical plane on a smooth horizontal table. The upper ends, $A$ and $B$, are connected by a light string, $ADB$, upon which a heavy ring can slide freely. Show that in equilibrium the height of $D$ above the table will be three-fourths that of $A$ or $B$.

21. $ABCD$ . . . is a closed polygon formed of any number of bars jointed together, and is in equilibrium under the action of forces acting at right angles to the bars at their middle points. Show that the actions at the joints are all equal.

If perpendiculars to the actions at $B$ and $C$ meet in $O$, $OBC$ may be taken as a triangle of forces for the rod $BC$ (turned through 90°). Thence show that, if the forces are proportional to the sides on which they act, the polygon can be inscribed in a circle.

22. Two equal uniform spheres of weight $W$ and radius $a$ rest in a spherical cup of radius $r$. Find the resistance $R$ between either sphere and the cup, and the pressure $P$ between the spheres.
$$R = \frac{r-a}{\sqrt{(r^2 - 2ar)}}W; \quad P = \frac{a}{\sqrt{(r^2 - 2ar)}}W.$$

23. A string, 21 inches long, is fastened to two nails 14 inches distant in a horizontal line, and weights $P$ and $Q$ are knotted to it at points 8 and 6 inches, respectively, from the two ends. Determine the ratio of $P$ to $Q$, so that in equilibrium the intermediate portion of the string shall be horizontal. $P : Q = 3 : 11$.

24. Two equal, uniform rods, each of length $2b$, are jointed

together at one end of each, and rest in equilibrium on a smooth cylinder with horizontal axis and radius $a$. Show that the angle $\theta$ which each rod makes with the horizontal is determined by

$$a \sin \theta = b \cos^2 \theta.$$

25. A uniform cylindrical shell of radius $c$ without a bottom stands on a horizontal plane, and two smooth spheres with radii $a$ and $b$, such that $a + b > c$, are placed within it. Show that the cylinder will not upset if the ratio of its weight to that of the upper sphere exceeds $2c - a - b : c$.

26. Two equal uniform planks, of length $b$ and weight $P$, are attached together and to two fixed points in a horizontal line by smooth hinges so that the angle each makes with the vertical is $\theta$. A sphere, of weight $W$ and radius $a$, rests between them. Find the tension on the lower hinge.

$$T = W\frac{b - a \cot \theta}{2b \sin \theta \cos \theta} + \frac{P \tan \theta}{2}.$$

27. A crane is formed of a vertical post fixed in the ground, with the part $AB$ 15 feet long above the ground at $A$, a horizontal bar $BD$ 12 feet long, jointed to the post at $B$, and a strut jointed to the post at $A$ and to $BD$ at a point $C$ 8 feet distant from $B$. At the end $D$ hangs a weight of 10 tons. Find the action at the joints $C$ and $B$. 17 tons; 9.434 tons.

## CHAPTER IV.

## PARALLEL FORCES AND CENTRES OF FORCE.

### VIII.

### Resultant of Three Parallel Forces.

**140.** We shall in this chapter consider systems of forces whose lines of action are all parallel but not in a single plane, the most familiar instances of which are afforded by the action of gravity upon different bodies or the parts of the same body.

It is convenient, in this case, to assume in the diagrams that the lines of action are perpendicular to the plane of the paper, and to suppose them to act at the points where the lines of action pierce the plane of the paper. We have seen that the resultant of two parallel forces is, in general, a force equal to their algebraic sum and acting in a line parallel to their lines of action. It evidently follows that the resultant of three parallel forces acting in lines perpendicular to the plane of the diagram is also a force equal to their algebraic sum, acting in a line perpendicular to the same plane. The point in which this resultant line of action pierces the plane is called *the centre of the parallel forces.*

**141.** The position of this centre depends upon the ratios of the given forces. In Fig. 50, let the forces $P_1$, $P_2$ and $P_3$, which are in the ratios $l : m : n$, act at the points $A$, $B$ and $C$ respectively, in lines perpendicular to the plane of the diagram. By Art. 85 the resultant of $P_1$ and $P_2$ is the force $P_1 + P_2$ acting at the point $F$ in which $AB$ is cut inversely in the ratio $l : m$, so that

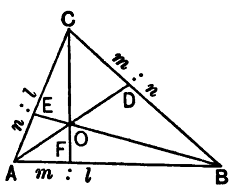


FIG. 50.

$$AF : FB = m : l,$$

as represented in the diagram. Join $CF$, then the resultant of this force and $P_2$ is the force $R = P_1 + P_2 + P_3$, acting at the point $O$ where $CF$ is cut inversely in the ratio $P_3 : P_1 + P_2$, or $n : m + l$, that is to say, so that

$$CO : OF = m + l : n.$$

**142.** In this construction for the centre of the three parallel forces $P_1$, $P_2$ and $P_3$, we shall arrive at the same point $O$ if we take the forces in any other order. Thus, if $BC$ be cut at $D$ inversely in the ratio of the forces $P_2$ and $P_3$, that is, in the ratio $n : m$, $O$ will be a point of the line $AD$. Hence $O$ may be found as the intersection of $CF$ and $AD$. When this is done and $O$ joined to $B$, it is readily seen that the areas of the triangles $AOC$, $COB$ are in the ratio $m : l$, and that the areas $AOC$, $BOA$ are in the ratio $m : n$. Thus the triangle $ABC$ is divided into three parts bearing the ratios $l : m : n$; and, if $BO$ be produced to $E$, $AC$ is cut in the ratio $n : l$, as indicated in the diagram.

The point $O$, thus determined by means of the ratios $l : m : n$, is also called *the centre of gravity* of three particles having these ratios and situated at $A$, $B$ and $C$ respectively: for, supposing the plane of the diagram horizontal, the forces may be taken as the weights of these particles, and $O$ is the point at which the resultant acts; or, as it is sometimes expressed, the point at which the total weight may be regarded as concentrated.

## Forces in Opposite Directions.

**143.** Parallel forces in opposite directions are called *unlike* parallel forces. If $P_1$ and $P_2$ are unlike, the numbers $l$ and $m$ have opposite signs, and we have seen in Art. 88 that the resultant cuts the transverse line produced in the inverse ratio of the weights; that is, on the side of the greater weight, and so that the whole line is to the part produced as the greater force is to the less. The construction for three forces is similar to that of Fig. 50, but the point $F$ falls on $AB$ produced, and $O$ is

found outside of the triangle $ABC$. When two of the three given forces have opposite directions, the third will be in the direction of one of them, and therefore one of the three points $D$, $E$ and $F$ will be on a side of the triangle. We may, in this case, take two of the numbers $l$, $m$ and $n$ as positive and the other as negative.

In Fig. 51, we consider the special case in which $m$ and $n$ are positive, and $l = -m$, so that the forces $P_1$ and $P_2$ form a couple in the plane passing through the line $AB$, and perpendicular to the plane of the diagram. The point $D$ will now be upon $CB$, $E$ will be upon $AC$ produced, and $F$ will be at an infinite distance on $AB$, so that $CO$ is parallel to $AB$. The resultant acting at $O$ is the algebraic sum of the forces, which is in this case $P_3$, because $P_1 = -P_2$.

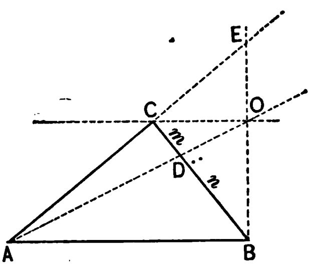

FIG. 51.

## Couples in Parallel Planes.

**144.** If we reverse the direction of this resultant, we shall have four forces in equilibrium; namely, $-P_2$ acting at $A$, $P_2$ acting at $B$, $P_3$ acting at $C$, and $-P_3$ at $O$. These constitute two couples in parallel planes, cutting the plane of the diagram in $AB$ and $CO$ respectively. By similar triangles, we have

$$CO : AB = m : n = P_2 : P_3;$$

therefore $$P_3 \times CO = P_2 \times AB;$$

that is to say, the moments of these couples are equal. Since the forces at $B$ and $C$ have the same direction, inspection of the figure shows that these couples tend to turn a rigid body about a perpendicular to $AB$ and $CO$ in opposite directions. Thus we

have proved that couples having equal moments in parallel planes are equivalent.

The construction in Fig. 51 is in fact, like that of Art. 101, the composition of a force and a couple, and the effect is, as in Fig. 36, not to change the magnitude or direction of the force, but to shift the line of action.

## Case in which the Resultant is a Couple.

**145.** A special case arises when the algebraic sum of the three forces is zero, so that $l + m + n = 0$. Let us suppose $m$ and $n$ to have the same sign; then, in Fig. 52, the point $D$ is upon the side $BC$ of the triangle $ABC$. The resultant of $P_2$ and $P_3$ is the force $P_2 + P_3$ acting at $D$. Now $P_1 = -(P_2 + P_3)$ by hypothesis; hence this force forms with $P_1$ a couple. Thus the resultant is in this case a couple in the plane passing through $AD$ and the line of action of $P_1$, and having for its moment $P_1 \times AD$.

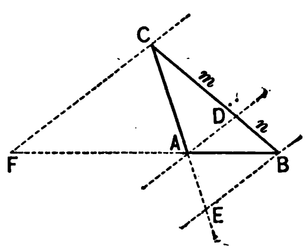


FIG. 52.

In this case, the point $F$ will lie upon $AB$ produced, and $E$ upon $CA$ produced, as in the figure, and we might equally well take for the resultant the couple $P_3 \times CF$, or the couple $P_2 \times BE$. In fact the three couples have equal moments and act in parallel planes, and are therefore equivalent by Art 144.

## Composition of Couples in Intersecting Planes.

**146.** The construction given above may be regarded as the composition of couples in planes which are not parallel. For, since $P_1 = -(P_2 + P_3)$, this force may be separated into parts equal and opposite respectively to $P_2$ and $P_3$. Therefore the given forces are equivalent to four forces forming two couples; namely, $P_2$ acting at $B$ with $-P_2$ at $A$, and $P_3$ acting at $C$ with

$-P_3$ at $A$. These two couples act in planes perpendicular to that of the diagram and intersecting it in $AB$ and $AC$, and their moments are $P_2 \times AB$ and $P_3 \times AC$, respectively. The construction therefore shows that the resultant of these couples is the couple $(P_2 + P_3)AD$ in a plane passing through $AD$ and the intersection of the planes of the given couples. The given couples and the resultant couple are here proportional to $mAB$, $nAC$ and $(m + n)AD$; hence, comparing the construction with that of Art. 67 for the resultant of two forces, it appears that the magnitude of the resultant and the parts into which the diedral angle between the planes is divided are precisely the same as in the case of two forces and the parts into which the plane angle between their lines of action is divided.

## Resolution of a Force into Parallel Components.

**147.** A given force $R$ can be resolved into components acting in any three given lines parallel to its line of action. Let the force act at $O$, and let the given lines intersect a plane through $O$ perpendicular to the line of action in $A$, $B$ and $C$. If these points are in a straight line, the problem is not determinate; but if they form the vertices of a triangle, the components have definite values.

First, suppose $O$ to lie within this triangle as in Fig. 50. Join $OA$, $OB$ and $OC$, and let $l$, $m$ and $n$ be three positive numbers proportional to the areas of the triangle $BOC$, $COA$, $AOB$; then, by Art. 142, $R$ acting at $O$ is the resultant of three forces proportional to $l$, $m$ and $n$, and having $R$ for their sum. Hence the required components are

$$\frac{lR}{l+m+n}, \quad \frac{mR}{l+m+n}, \quad \frac{nR}{l+m+n}.$$

The ratios of $l$, $m$ and $n$ may of course also be determined by means of the segments of the sides of the triangle as indicated in Fig. 50.

**148.** If $O$ falls upon one side of the triangle, the component at the opposite vertex vanishes, and the problem reduces to that of Art. 87 for determining two components coplanar with the given force. If $O$ falls outside of the triangle, one or two of the triangles whose vertices are at $O$ must be taken as negative; for example, in Fig. 51 the value of $l$ is negative and the component at $A$ has a direction opposite to that of $R$.

As an illustration, we may suppose $A$, $B$ and $C$ to be the points in a three-legged table directly over the three feet. Then, when a weight $W$ is placed at $O$ upon the table, the components are the pressures produced by $W$ at the three feet and resisted by the floor. If $O$ lies beyond the line $BC$, as in Fig. 51, the direction of the component at $A$ is reversed, and, since it now acts upward, the foot must be held down to produce equilibrium.

## Moment of a Force about an Axis.

**149.** In Art. 91, the action of a couple upon a lamina in its plane is explained as a tendency to turn it in its own plane, that is to say, about an axis perpendicular to the plane of the couple. Furthermore, in Art. 93, we have seen that if the axis, say through $B$, Fig. 29, is fixed, while a single force acts on the lamina at $A$, the resistance of the axis, together with the given force, constitutes a couple, and the moment of this couple is called the moment of the force about the point $B$, or more properly about the axis through $B$ perpendicular to the plane of the diagram. In Fig. 53 we represent the axis $MN$ passing through $B$ as in the plane of the diagram, while the force at $A$ is supposed to act perpendicularly to the plane of the diagram. The arm $AB$ of the moment is perpendicular both to the line of action and to the axis. If the force $P$ be transferred to the

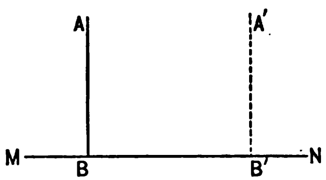


FIG. 53.

point $A'$ equally distant with $A$ from the axis $MN$, it will have

the same moment; and since we have now seen, in Art. 144, that couples of the same moment in parallel planes are equivalent, it is not necessary to specify the point $B$ at which the arm meets the axis. We can therefore compare directly, and employ in the theorem of moments, the moments of forces about a given axis, although the lines of action are not in one plane. It is only necessary for our present definition of a moment that the line of action and the axis (though not intersecting) should be *in directions at right angles to one another.*

**150.** In the case of parallel forces, the diagrams being drawn, as in the present section, in a plane perpendicular to the lines of action, the forces have moments about all straight lines in the plane of the diagram, and we can apply the theorem of moments with respect to any such line as axis. Thus, in Fig. 50, the moments of $P_1$ and of $P_2$ about the axis $AB$ are both zero; hence, by the theorem of moments, the moment of $P_3$ about $AB$ is the moment of the resultant at $O$ about the same axis. Accordingly, the perpendiculars from $C$ and $O$ are inversely as $P_3$ is to $R$, which agrees with the result in Art. 142, since the triangles $ABC$ and $AOB$ are proportional to their altitudes.

## The Centre of Parallel Forces.

**151.** The graphical determination of the resultant of any number of parallel forces acting at given points in a plane perpendicular to the direction of the forces is an extension of the process of Art. 141, involving successive applications of the operation of cutting a given line in a given ratio. At each step we combine two forces of the system into one, and the order in which the forces are taken is arbitrary. Thus, if four equal forces of magnitude $P$ act at $A$, $B$, $C$ and $D$, Fig. 54, we may determine the resultant of the forces at $A$ and $B$ by bisecting $AB$; and then (instead of combining $2P$ acting at this point with one of the other forces, which would require a trisection) combine the forces at $C$ and $D$ in like manner at the middle point of $CD$. The system is thus reduced to two forces, each

equal to $2P$, and we have finally to combine them by a third bisection.

If we change the order in which the points are taken, we shall arrive at the same final position; thus it appears that *the lines bisecting pairs of opposite sides or diagonals of a plane quadrilateral bisect each other.*

152. But when the positions of the points of application are given by means of their distances from given lines, it is generally more convenient to determine the centre of the parallel forces by means of their moments with respect to the given lines. The algebraic sum of the forces gives the magnitude of the resultant force $R$; and, supposing this not to be zero (in which case the resultant is a couple), the resultant moment of the system about any line is the moment of the resultant, and therefore determines the distance from that line at which it acts.

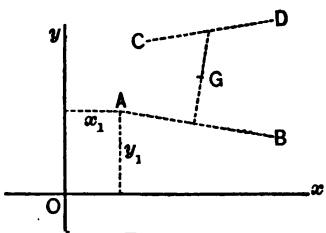


FIG. 54.

Thus, supposing the forces referred to rectangular axes, the moment about the axis of $y$ of the force $P_1$ acting at $(x_1, y_1)$, Fig. 54, is $P_1x_1$, that of $P_2$ acting at $(x_2, y_2)$ is $P_2x_2$, and so on, for all the forces of the system. The algebraic sum of these moments is equal to the moment of $R$ acting at the centre of force $(\bar{x}, \bar{y})$, which is $R\bar{x}$, and therefore determines the distance of that point from the axis of $y$. Using, in like manner, moments about the axis of $x$, we have for the determination of $R$ and the coordinates of its point of action:

$$R = P_1 + P_2 + \dots\dots + P_n = \Sigma P, \quad . \quad . \quad . \quad (1)$$

$$R\bar{x} = P_1x_1 + P_2x_2 + \dots + P_nx_n = \Sigma Px, \quad . \quad . \quad (2)$$

$$R\bar{y} = P_1y_1 + P_2y_2 + \dots + P_ny_n = \Sigma Py. \quad . \quad . \quad (3)$$

## Case in which R = 0.

153. If the given forces have not all the same direction, some of the terms in $\Sigma P$ are negative and the sign of $R$ determines the direction of the resultant, and together with that of the total moments $\Sigma xP$ and $\Sigma yP$ the signs of $\bar{x}$ and $\bar{y}$ in equations (2) and (3). When $R = 0$, and $\Sigma Px$ or $\Sigma Py$ is not zero, at least one of the quantities $\bar{x}$ or $\bar{y}$ is infinite. The resultant is now not a force, but a couple which may be determined as follows: Assume two equal and opposite forces $Q$ and $-Q$ acting at the origin in a line perpendicular to the plane of the diagram. The addition of these forces to the system will not change the resultant. Now, since $Q$ acting at the origin has no moment about either axis, the moments of the system consisting of the given forces and the force $Q$ are still $\Sigma Px$ and $\Sigma Py$, and the resultant of this system (which includes the force $Q$ but *not* the force $-Q$) is $Q$, because $\Sigma P$ is by hypothesis zero. Hence the resultant of this system is the force $Q$ acting at $(x', y')$ when

$$Qx' = \Sigma Px \quad \text{and} \quad Qy' = \Sigma Py.$$

Therefore the resultant of the given system is the couple formed by the force $Q$ acting at $(x', y')$ and the force $-Q$ acting at the origin.

This couple acts in a plane perpendicular to that of the paper intersecting it in the line joining $(x', y')$ to the origin, and its arm is the distance of $(x', y')$ from the origin. Hence, denoting its moment by $H$ and the inclination of its plane to the axis of $x$ by $\theta$, we have

$$H = Q\sqrt{(x'^2 + y'^2)} = \sqrt{[(\Sigma Px)^2 + (\Sigma Py)^2]}$$

and

$$\tan\theta = \frac{y'}{x'} = \frac{\Sigma Py}{\Sigma Px}.$$

## Conditions of Equilibrium.

**154.** The system of parallel forces is in equilibrium if the resultant force and the resultant moments about each axis all vanish, that is, the conditions of equilibrium are

$$\Sigma P = 0, \qquad \Sigma xP = 0, \qquad \Sigma yP = 0.$$

Thus, for forces whose lines of action are all parallel, as well as for forces whose lines of action are all in one plane, there are *three* independent conditions of equilibrium necessary. But, in this case, only one condition can be derived from the equality of forces in a given direction, and two must be derived from the principle of moments.

It is evident that these two conditions may involve moments about any two axes, but to be independent conditions, these axes must not be parallel. For, if $R = 0$, the resultant might be a couple, and if the moment about a given line vanished, it might still be a couple in a plane passing through the given line; and, in that case, the moment about any parallel line would necessarily vanish since couples in parallel planes are equivalent.

## The Centre of Gravity of $n$ Particles.

**155.** The point $(\bar{x}, \bar{y})$ of Art. 152 or centre of parallel forces is, when all the forces have the same sign, also called *the centre of gravity* of particles whose weights are $P_1, P_2, \ldots P_n$, situated at the given points $(x_1, y_1)$, $(x_2, y_2)$, etc. For, if we suppose the plane of the diagram horizontal, the forces may be regarded as the weights of these particles.

It will be noticed that if the particles are all equal, the centre of gravity $(\bar{x}, \bar{y})$ is *the centre of position* of the given points, mentioned in Art. 69, of which the distance from any plane is *the arithmetical mean* or average of the distance of the given points. Thus the process in Art. 151 is that of finding the centre of gravity of four equal particles situated in a plane.

**156.** In the general case, the weights being unequal, the

values of $\bar{x}$ and $\bar{y}$ determined by the equations of Art. 152, namely,

$$\bar{x} = \frac{\Sigma Px}{\Sigma P} \quad \text{and} \quad \bar{y} = \frac{\Sigma Py}{\Sigma P},$$

are called *the weighted means* of the given values of $x$ and of $y$ respectively. The particles are here all supposed to be in one horizontal plane, and the algebraic sum $\Sigma Pp$ of their moments about *any* axis passing through the centre of gravity is zero; so that, if rigidly connected to a lamina without weight, they would balance about such an axis. Moreover, this remains true also when the plane is turned through any angle $\theta$, for the arm $p$ of each moment in $\Sigma Pp$ is thus changed to $p \cos \theta$; therefore the resultant moment becomes $\cos \theta \,.\, \Sigma Pp$, and is equal to zero because $\Sigma Pp = 0$. Hence the centre of gravity is the point through which the resultant of the weight passes, so long as the particles retain their relative position, irrespective of the direction of gravity.

**157.** In many cases considerations of symmetry make the position of the centre of gravity obvious. For example, the centre of gravity of two equal particles at opposite vertices of a parallelogram, together with two other equal particles at the remaining vertices, is the intersection of the diagonals. The centre of gravity of equal particles at the vertices of a regular polygon is the centre of the figure.

In other cases, these considerations will aid in determining the centre of gravity. For example, the centre of gravity of four equal particles at the vertices $B$, $C$, $D$, $E$ of the regular pentagon $ABCDE$, Fig. 55, is the point of application of the resultant of a force equal to the weight of five such particles at the centre $O$ and an upward force equal to the weight of one such particle at $A$. Hence the centre of gravity $G$ divides $AO$ *externally* in the ratio 1 : 5, that is, $G$ is on $AO$ produced at a distance $OG = \frac{1}{4}AO$.

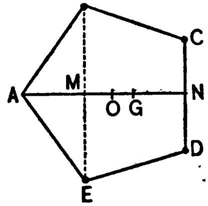

FIG. 55.

Since it is otherwise obvious that the centre of gravity $G$ is mid-

way between the points where $CD$ and $BE$ cut the diameter through $A$, it is thus proved that the difference between the distances $OM$ and $ON$ of these lines from the centre is one half of the radius. This could not be so easily proved in any other manner.

## EXAMPLES. VIII.

1. The legs of a right triangle are 3 and 4 feet long; a force of 1 pound acts at the right angle, a force of 2 pounds at the greater acute angle, and a force of 3 pounds at the smaller, all in the same direction. Construct the centre of the forces and find its distance from the legs. 2 feet, and 1 foot.

2. In example 1 reverse the direction of the force of 2 pounds and find the resultant.
2 pounds at a point distant 6 and 3 feet respectively.

3. In Ex. 1 reverse the force of 3 pounds and find the resultant. A couple of moment 6 $\sqrt{5}$ pounds-feet.

4. Show that, if parallel forces proportional to the tangents of the angles act at the vertices of a triangle, their resultant acts at the orthocentre.

5. A table whose top is in the form of a right-angled isosceles triangle, the equal sides of which are 3 feet in length, is supported by three vertical legs placed at the corners; a weight of 20 pounds is placed on the table at a point distant 15 inches from each of the equal sides. Find the resultant pressure on each leg.
$8\frac{1}{3}$; $8\frac{1}{3}$; $3\frac{1}{3}$.

6. A weight $W$ is supported by a tripod of equal legs whose feet form the vertices of the horizontal triangle $ABC$. Show that the vertical components of the compressions of the legs are in the ratios

$$\sin 2A : \sin 2B : \sin 2C.$$

7. The three equidistant feet of a circular table are vertically beneath the rim. Assuming that the weight of the table acts at the centre, show that a weight less than that of the table cannot upset it.

8. A uniform circular disk is placed on a triangular frame of light rods which is supported in a horizontal position at its vertices, the disk projecting equally over each rod. Show that the upward resistances at the supports are proportional to the lengths of the opposite rods.

9. Unlike parallel forces act at the vertices of a triangle and are proportional to the opposite sides. Show that the centre of the forces is the centre of one of the *escribed* circles (touching one side and two sides produced).

10. Four like parallel forces act at the vertices of the quadrilateral $ABCD$ in a plane. The forces at the several points are proportional each to the triangle whose vertices are the other three points. Show that the centre of the forces is the intersection of $AC$ and $BD$ if they do intersect. How must the forces act that this may be true when it is necessary to produce one of these lines to intersect the other?

11. A force of 2 pounds acts perpendicularly to the plane of $xy$ at $(1, -1)$, and parallel forces of 3 pounds at $(0, 4)$, 1 pound at $(-1, 3)$ and $-4$ pounds, that is, 4 pounds in the opposite direction, at $(1, 2)$. Find the resultant.

2 pounds acting at $(-1\frac{1}{2}, 2\frac{1}{2})$.

12. Reverse the direction of the force of 1 pound in example 11, and find the resultant.

A couple which may be represented by the force 1 at the origin and $-1$ at the point $(1, 1)$.

13. Show that the centre of gravity of particles in a plane of weights 1, 2, 4 and 8 can be found graphically by three bisections and one other operation.

14. Three uniform rods of like material but of different lengths form the sides of a triangle. Show that the centre of gravity is the centre of the circle inscribed in the triangle formed by joining the middle points.

15. In example 8, if the disk just touches the rods, show that the pressures at the points of contact are equal to those at the opposite supports respectively.

## IX.

## The Centre of Uniform Pressure on a Plane Surface.

**158.** When a solid body is urged by any external force against a fixed plane surface, the resistance offered by the surface takes place at a limited number of points of contact; so that the mode in which the pressure resisted is distributed depends upon the accuracy with which the surfaces in contact fit each other and the degree of rigidity of the materials.

On the other hand, when a liquid or gaseous body presses upon a surface, the resistance takes place at all points and is distributed over the surface in some regular and continuous manner. When the distribution is such that two portions of equal area taken in any parts of the surface sustain equal pressures, the pressure on the surface is said to be *uniformly distributed.*

The pressures on the several parts of the area all act in lines perpendicular to the plane surface, and therefore parallel to one another; hence the total or resultant pressure is their sum. The pressure upon a unit of area is taken as the measure of the intensity of the pressure; for example, the number of pounds sustained by a square inch of the surface. Denoting this by $p$, and the number of units of area by $A$, the total or resultant pressure is $P = pA$, and the point at which this resultant acts is called *the centre of pressure.*

The case of uniform pressure is realized by a horizontal surface sustaining a uniform depth of water, or by a plane surface in any direction sustaining the pressure of an elastic fluid like steam.

**159.** To find the centre of a uniform pressure $p$ upon a given surface of area $A$, we refer the surface to rectangular coordinate axes, and select some convenient element of area $dA$. The pressure upon an element is then $pdA$. We have next to find the moment of this element of pressure about each of the axes; then

in the formulæ of Art. 152 we have only to replace summation by integration. For example, to find the centre of uniform pressure on a quadrant of a circle whose radius is $a$, we refer it, as in Fig. 56, to its bounding radii as axes. The total pressure is known, because the area is known; it is $P = \frac{1}{4}\pi p a^2$. In finding the moment about the axis of $y$, it is convenient to take the element of area $dA = ydx$ (as represented in the figure), because every point of this element is at the same distance, namely, $x$, from the axis of $y$. Then the element of moment about the axis of $y$ is the pressure upon the element multiplied by the arm $x$. The pressure on the element is $pdA$; hence the element of moment is

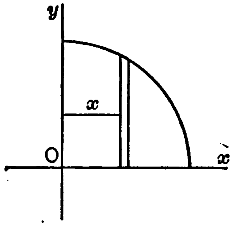

FIG. 56.

$$dM = pxdA,$$

and the moment, $M$, is the integral of this expression between proper limits.

160. In the present example, the equation of the circular boundary of the area is $x^2 + y^2 = a^2$, whence

$$y = \sqrt{(a^2 - x^2)}.$$

Substituting in the expression for the element of the moment about the axis of $y$, we have

$$dM = px\sqrt{(a^2 - x^2)}dx.$$

The figure shows the limits of integration to be o and $a$; hence

$$M = p\int_0^a (a^2 - x^2)^{\frac{1}{2}} x dx = -\frac{p}{3}(a^2 - x^2)^{\frac{3}{2}}\Big]_0^a = \frac{pa^3}{3}.$$

Denoting the abscissa of the centre of pressure by $\bar{x}$, the moment of the resultant, or total pressure $P$, is $P\bar{x}$, and this must equal the total moment about the axis of $y$. Hence

$$P\bar{x} = M;$$

and, substituting the values of $P$ and $M$ in this example,

$$\bar{x} . \frac{\pi p a^2}{4} = \frac{p a^3}{3}; \qquad \text{whence} \qquad \bar{x} = \frac{4a}{3\pi}.$$

**161.** The value of $\bar{y}$, the ordinate of the centre of pressure, may be found in the same way, using the element of area $x dy$, and in this example the value will be the same as that of $\bar{x}$, because $x$ and $y$ are, in the equation of the circle, interchangeable. But it may also be found by employing the element of area used in Fig. 56, so that $dP = py dx$ still denotes the element of pressure. It must be noticed, however, that the points of this element are *not* all at the same distance from the axis of $x$; hence, denoting the moment about the axis of $x$ by $M_x$, the arm of the elementary moment $dM_x$ is the *average value* of the distances of the several points. Since the element has a uniform breadth, this average value is obviously $\frac{1}{2}y$. Therefore

$$dM_x = \tfrac{1}{2} p y^2 dx;$$

substituting the value of $y^2$ and integrating,

$$M_x = \tfrac{1}{2} p \int_0^a (a^2 - x^2) dx = \tfrac{1}{2} p (a^2 x - \tfrac{1}{3} x^3) \Big]_0^a = \tfrac{1}{3} p a^3.$$

Finally, putting $P\bar{y}$ for $M_x$, we find $\bar{y} = \dfrac{4a}{3\pi}$.

## The Centre of Gravity of an Area

**162.** A *uniform heavy lamina* is a thin plate of uniform thickness and material, so that the weight of a unit of area taken in

any part of it is equal to a constant $w$. When such a lamina is in a horizontal plane, the resultant of its weight is evidently the same as that of a uniform pressure $w$ upon the area of the lamina. Hence the centre of uniform pressure is also *the centre of gravity* of the lamina; that is, the point at which, if the whole weight were concentrated, it would produce the effect of the resultant weight. The factor $w$ does not affect the position of this point, which is therefore called *the centre of gravity of the area*, and also sometimes *the centroid of the area.*

In finding its position, as in Arts. 160 and 161, $w$, which takes the place of $p$, is generally put equal to unity, and the moments about the axes are then called *the moments of the area*, or *the statical moments* of the area.

**163.** The centroid of a plane area corresponds to *the centre of position* of all its points, so to speak; for equal weight is given to all its points, that is, to all equal small areas. Accordingly, its distance from any straight line is regarded as *the average distance* of all its points. The position of the centroid is often obvious from considerations of symmetry. For example, the centroid of a circle, of an ellipse, or of a regular polygon is the geometric centre; that of a parallelogram is the intersection of the diagonals.

Again, whenever there is an axis which bisects a set of elements of uniform width which make up the area, the centroid must be upon this axis. For the centre of gravity of each element is upon this axis, so that the weight of the whole area has the same resultant as that of a number of particles situated upon the axis, and there is no reason why this resultant should be on one side of the axis rather than the other. In particular, the centroid of a triangle, Fig. 57, is thus seen to be upon a medial line; and, since for the same reason it is on each of the other medial lines, it is at the point where the medial lines meet. Comparing with Art. 141, it is the position of $O$ when $l = m = n$. Hence the centre of gravity of a triangle

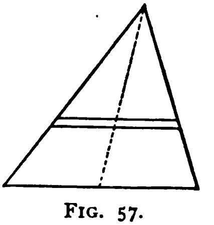

FIG. 57.

is the same as that of three equal particles at its vertices. That is to say, it is on a medial line *at a distance of two thirds of the medial line from the vertex*, and its perpendicular distance from the base is *one third of the altitude.*

**164.** When the centre of gravity of a figure is known as well as its area, its statical moment about any axis in the plane may be found without resorting to integration. The centre of gravity of an area made up of parts which are thus known is then readily determined. For example, given the trapezoid $ABCD$, Fig. 58, whose parallel sides are $a$ and $b$ and whose altitude is $h$. By Art. 162, the centre of gravity is on the line joining the middle points of $AB$ and $CD$. To find its distance from the base $b$, we divide the area into two triangles by the diagonal $BD$, and take statical moments about $AB$. The moment of $ABD$ is its area $\frac{1}{2}bh$ multiplied by the distance of its centre of gravity from $AB$, which, by Art. 163, is $\frac{1}{3}h$. In like manner, the moment of $DCB$ is $\frac{1}{2}ah \times \frac{2}{3}h$. Denoting the required distance by $\bar{x}$, the moment of the whole figure is the total area multiplied by $\bar{x}$. Hence

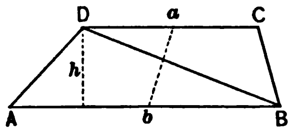


FIG. 58.

$$\tfrac{1}{2}h(a+b)\bar{x} = \tfrac{1}{2}bh \cdot \tfrac{1}{3}h + \tfrac{1}{2}ah \cdot \tfrac{2}{3}h;$$

therefore

$$\bar{x} = \frac{b+2a}{3(a+b)}h.$$

**165.** In like manner, the given area may be the difference of areas whose centres of gravity are known. For example, from a square whose side is $2a$, Fig. 59, an equilateral triangle constructed on the side $AB$ is removed, and this triangle is turned about $AB$ into the position shown in the diagram; it is required to find the centre of gravity of the figure thus formed. The centre of gravity is upon the axis of symmetry $CD$ bisecting $AB$, and we have only to find its distance $x$ from $AB$. The

moment of the square about $BA$ is $4a^2 . a = 4a^3$. The area of the triangle is $a^2\sqrt{3}$, since the altitude is $a\sqrt{3}$; the centre of gravity is at a distance $\frac{1}{3}\sqrt{3}a$, therefore the moment of the triangle is $a^3$. To find the moment of the figure in its present position, we must subtract from the moment of the square the moment of the triangle removed, and also algebraically add the moment of the triangle in its new position, which is negative because it is on the other side of $AB$. This total moment is to be put equal to the total area multiplied by the arm $x$. Thus

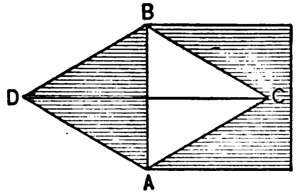


FIG. 59.

$$4a^2x = 4a^3 - a^3 - a^3,$$

whence

$$x = \frac{1}{2}a.$$

## The Centre of Gravity of a Uniform Curve.

**166.** The weight of a thin uniform rod, or wire, having the form of a plane curve may be regarded as uniformly distributed along the length of a mathematical line. Supposing the plane horizontal, the resultant weight pierces the plane in a point, which is called *the centre of gravity of the arc*, but which generally will not be situated upon the curve. Omitting the factor expressing the weight of a unit of length of the rod, the moment about any axis is called the statical moment of the given curve (whose length we denote by $s$); and this is to be put equal to the moment about the same axis of $s$ concentrated at the point $(\bar{x}, \bar{y})$.

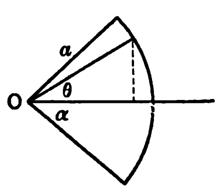


FIG. 60.

As an illustration, let us determine the centre of gravity of a circular arc of radius $a$ and angular measure $2\alpha$. By symmetry, the centre of gravity is upon the radius bisecting the arc. Taking this radius as the axis of $x$, and the centre as origin, and denoting the angular

distance of the element $ds$ from the axis by $\theta$, we have, since $x = a\cos\theta$ and $ds = ad\theta$,

$$\bar{x}s = a^2\int_{-\alpha}^{\alpha}\cos\theta\,d\theta = 2a^2\sin\alpha.$$

Therefore, since $s = 2a\alpha$, $\bar{x} = a\dfrac{\sin\alpha}{\alpha}$.

In particular, when $\alpha = \dfrac{\pi}{2}$, this gives $\bar{x} = \dfrac{2a}{\pi}$ for the distance from the centre of the centre of gravity of a semi-circle. Again, when $\alpha = \pi$, we find $\bar{x} = 0$, which is obviously correct for the complete circle.

## Employment of Polar Coordinates.

**167.** When the boundary of an area is given by its equation in polar coordinates, the centre of gravity is still referred to rectangular axes, the axis of $x$ coinciding with the initial line. It is necessary to use a polar element of area, and employing the usual element

$$dA = \tfrac{1}{2}r^2d\theta,$$

which is supposed to lie wholly within the area, we have to find its moments about the axes. Since this element (see Fig. 61) is triangular in form, its centre of gravity is at a distance $\frac{2}{3}r$ from the pole, that is, at the point whose rectangular coordinates are $\frac{2}{3}r\cos\theta$ and $\frac{2}{3}r\sin\theta$. Hence the elements of moment about the axes of $y$ and $x$, respectively, are

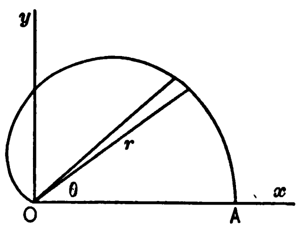

Fig. 61.

$$dM_x = \tfrac{1}{3}r^3\sin\theta\,d\theta,\quad dM_y = \tfrac{1}{3}r^3\cos\theta\,d\theta.$$

**168.** For example, let it be required to find the centre of gravity of the area of the half cardioid, Fig. 61, whose polar equation is

$$r = a(1 + \cos\theta).$$

Substituting this value of $r$, we have for the elements in terms of $\theta$

$$dA = \tfrac{1}{2}a^2(1 + \cos\theta)^2 d\theta,$$
$$dM_y = \tfrac{1}{3}a^3(1 + \cos\theta)^3 \cos\theta\, d\theta,$$
$$dM_x = \tfrac{1}{3}a^3(1 + \cos\theta)^3 \sin\theta\, d\theta.$$

The figure shows that the limits of integration for $\theta$ are 0 and $\pi$. Hence,

$$A = \frac{a^2}{2}\int_0^\pi (1 + \cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)d\theta$$
$$= \frac{a^2}{2}(\pi + \tfrac{1}{2}\pi) = \frac{3\pi a^2}{4},$$

$$M_y = \frac{a^3}{3}\int_0^\pi (1 + \cos\theta)^3 \cos\theta\, d\theta$$
$$= \frac{a^3}{3}\int_0^\pi (\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta)d\theta$$
$$= \frac{a^3}{3}\left[\frac{3}{2} + \frac{3\cdot 1}{4\cdot 2}\right]\pi = \frac{5\pi a^3}{8},$$

$$M_x = \frac{a^3}{3}\int_0^\pi (1 + \cos\theta)^3 \sin\theta\, d\theta = -\frac{a^3}{3}\frac{(1 + \cos\theta)^4}{4}\Big]_0^\pi = \frac{4a^3}{3};$$

whence, since $\bar{x}A = M_y$ and $\bar{y}A = M_x$, we find

$$\bar{x} = \frac{5}{6}a, \qquad \bar{y} = \frac{16}{9\pi}a,$$

for the coordinates of the centre of gravity.

169. If we employ the ultimate polar element of area, $r\,dr\,d\theta$, which is situated anywhere within the area ($r$ and $\theta$ being now independent variables), the expressions for the area and the moments are the double integrals

$$A = \iint r\,dr\,d\theta,$$

$$\bar{x}A = \iint r^2 \cos\theta\, d\theta,$$

$$\bar{y}A = \iint r^2 \sin\theta\, d\theta.$$

Performing the $r$-integration first, as is usually most convenient, the limits are zero and the $r$ of the curve; and the results under the single integral sign are the elements given in Art. 167, in which $r$ represents a given function of $\theta$.

In like manner, the elements of moment for single integration given in Arts. 158 and 160 may be derived from the moments of the ultimate element of area $dy\,dx$, namely,

$$x\,dy\,dx \qquad \text{and} \qquad y\,dy\,dx,$$

by performing the $y$-integration between the limits zero and the ordinate of the given curve.

## The Theorems of Pappus.

**170.** The centre of gravity regarded as the point at the average distance has a useful application to surfaces and solids of revolution. Suppose an arc of a plane curve to revolve about an axis in its plane, but not crossing the arc, thus generating a surface of revolution. Taking the axis of revolution as the axis of $y$, every element $ds$ of the arc describes in the revolution a circle whose radius is $x$. The circumference of this circle, or path of $ds$, is $2\pi x$, hence it generates the element of surface $2\pi x ds$. Therefore the whole surface generated is the integral of this expression taken between the limits which define $s$. Denoting it by $S$, we have then $S = 2\pi \int x ds$; but the integral in this expression is the statical moment of $S$ about the axis of $y$, which is the value of $\bar{x}s$. Hence the surface of revolution is

$$S = 2\pi\bar{x}\,s,$$

in which $2\pi\bar{x}$ is the circumference described by the centre of gravity. Hence *the surface generated is equal to the product of the length of the arc and the path of its centre of gravity.*

**171.** Again, let a plane area revolve about an axis in its plane but not crossing its surface, thus generating a volume of revolu-

tion. Taking as before this axis as axis of $y$, every element of area $dA$ describes a circle whose radius is $x$ and circumference $2\pi x$. Hence it generates the element of volume $2\pi x dA$, and the whole volume generated is the integral of this expression taken with the same limits which define the area $A$. Denoting the volume by $V$, we have then $V = 2\pi \int x dA$, where the integral is the statical moment of $A$ with respect to the axis of $y$. The value of this moment is $\bar{x}A$, where $\bar{x}$ is the distance from the axis of $y$ of the centre of gravity of the area $A$. Hence

$$V = 2\pi\bar{x}A,$$

in which $2\pi\bar{x}$ is the circumference described by the centre of gravity. Therefore *the volume generated by the revolution of a plane area about an axis, in its plane and not crossing its surface, is the product of the area and the path of its centre of gravity.*

These theorems, sometimes called Guldin's Theorems, are properly the Theorems of Pappus, having been first given in the "Collection" of Pappus, a mathematician of Alexandria who flourished probably about 300 A.D.

**172.** It will be noticed that, in each of the theorems, "the path of the centre of gravity" is the average length of the path of the elements of the generating line or area as the case may be, just as the arm of the total moment is the average arm of the elements.* The useful applications are not to cases in which it would be necessary to find the centre of gravity by integration,†

* Pappus's theorems evidently apply to any part of a revolution, or to any motion in which the elements are always moving in a direction perpendicular to the plane of the generating figure, and in like directions. Compare with the description of an area by a straight line, Int. Calc., Art. 163, where the motion considered is the resolved part perpendicular to the generating line, as recorded by the wheel in Amsler's Planimeter.

† The method for a volume of revolution given in Int. Calc., Art. 136, involves precisely the integral we should employ in finding the statical moment about the axis of revolution, with the addition of the factor $2\pi$.

but to those in which the position of the centre of gravity and the area are known.

For example, if the circle of radius $a$, Fig. 62, revolve about the axis $AB$ at a distance $b$ from the centre, it generates a solid generally called an "anchor-ring." The centre of gravity being the centre of the circle, its path is $2\pi b$. Therefore the volume of the anchor-ring is

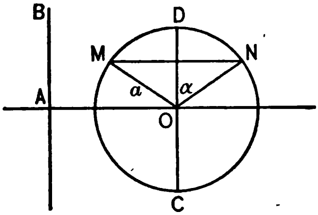


FIG. 62.

$$V = \pi a^2 \cdot 2\pi b = 2\pi^2 a^2 b.$$

Again, the centre is also the centre of gravity of the circumference which generates the surface of the anchor-ring. Hence we have for the surface

$$S = 2\pi a \cdot 2\pi b = 4\pi^2 ab.$$

**173.** The segment of the anchor-ring cut off by a plane perpendicular to the axis is generated in the revolution by a segment such as $MDN$ of the generating circle. The centre of gravity of this segment, and also that of the arc $MN$, is at the same distance $b$ from the axis of revolution. Denoting the half of the angle subtended by the chord $MN$ by $\alpha$, the length of the arc is $2a\alpha$. The area of the circular segment is the area of the sector $OMDN$, which is $a^2\alpha$, diminished by that of the triangle $OMN$, which is $a^2 \sin\alpha \cos\alpha$. Hence we have, for the volume and surface of the segment of the ring,

$$V = 2\pi ba^2(\alpha - \sin\alpha\cos\alpha) \qquad \text{and} \qquad S = 4\pi\alpha ab.$$

**174.** Another useful application of Pappus's theorems is the determination of the centre of gravity of the generating area, when the area itself and the volume generated are both supposed known. For example, if the semi-circle $CND$ in Fig. 62 revolves about the diameter $CD$ it will generate a sphere. Denoting the

distance of the centre of gravity of the semi-circle from the diameter by $\bar{x}$, we have by the second theorem, since the known volume of the sphere is $\frac{4}{3}\pi a^3$,

$$\tfrac{1}{2}\pi a^2 \,.\, 2\pi\bar{x} = \tfrac{4}{3}\pi a^3;$$

whence $\bar{x} = \dfrac{4a}{3\pi}$, agreeing with the result found by integration* in Art. 160.

In like manner, because the semi-circumference *DNC* generates the surface of the sphere, of which the value is $4\pi a^2$, we have, when $\bar{x}$ denotes the distance of the centre of gravity of the semi-circumference,

$$\pi a \,.\, 2\pi\bar{x} = 4\pi a^2;$$

whence $\bar{x} = \dfrac{2a}{\pi}$, agreeing with Art. 166.

**175.** The reason for the restriction that the axis of revolution must not cross the generating area is that the distance $x$, in the expression $xdA$ of the demonstration in Art. 171, has opposite signs for the portions of the area on opposite sides of the axis. Hence, in the result, the volumes generated by the two portions have to be taken with opposite algebraic signs, just as the statical moments of these parts have opposite signs. For example, let the circle in Fig. 63 revolve about the chord *AB*, which cuts off 120° of the circumference, and therefore bisects the radius *CD* to which it

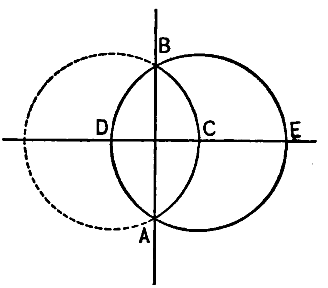


FIG. 63.

* If the volume had to be found by integration, it might be found directly, as mentioned in the preceding foot-note, by the same integration which is performed in Art. 160 to find the statical moment. But, in the process above, we are free to employ other methods of integrating for the volume.

is perpendicular. The centre of gravity of the circle is at a distance $\frac{1}{2}a$ from $AB$, and the theorem gives

$$V = \pi a^2 \,.\, \pi a = \pi^2 a^3,$$

which is *the difference* between the volumes generated by the segments $AEB$ and $ADB$ respectively.

Now since this is the same as the volume generated by the lune-shaped area $ACBE$, it will enable us to find the centre of gravity of that area. For the area of the lune, being the difference between the segments $AEB$ and $ACB$ or $ADB$, is

$$A = \tfrac{2}{3}\pi a^2 + \frac{\sqrt{3}}{4}a^2 - \left(\tfrac{1}{3}\pi a^2 - \frac{\sqrt{3}}{4}a^2\right) = \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)a^2.$$

Therefore, regarding the volume $V$ as generated by this area, and denoting the distance of its centre of gravity from $AB$ by $\bar{x}$, we have

$$\frac{2\pi + 3\sqrt{3}}{6}a^2 \,.\, 2\pi\bar{x} = \pi^2 a^3,$$

whence $\bar{x} = \dfrac{3\pi a}{2\pi + 3\sqrt{3}} = .821a$, so that the centre of gravity is at a distance $.321a$ from the centre of the circle.

## EXAMPLES. IX.

1. Find by integration the distance from the base of the centre of gravity of a triangle and of a trapezoid.

2. Find the centre of gravity of the area between the parabola $y^2 = \dfrac{b^2}{h}x$ and the double ordinate $2b$. $\bar{x} = \frac{3}{5}h.$

3. Find the ordinate of the centre of gravity of the upper half of the area in Ex. 2. $\bar{y} = \frac{3}{8}b.$

4. Find the centre of gravity of the area between the parabola in Ex. 2, the axis of $y$, and the perpendicular to it from the point $(h, b)$. $\bar{x} = \frac{3}{10}h;\ \bar{y} = \frac{3}{4}b.$

5. Find the centre of gravity of the area between the semi-cubical parabola $ay^2 = x^3$ and the double ordinate which corresponds to the abscissa $a$. $\bar{x} = \frac{5}{7}a.$

6. Find the distance of the centre of gravity of a sector of a circle from the centre of the circle, the radius being $a$ and the angle of the sector $2\alpha$.

$$\bar{x} = \frac{2a \sin \alpha}{3\alpha}.$$

7. Determine the centre of gravity of the area included between the parabola $y^2 = 4ax$ and the straight line $y = mx$.

$$\bar{x} = \frac{8a}{5m^2}; \; \bar{y} = \frac{2a}{m}.$$

8. A uniform wire is bent into the form of a circular arc and its two bounding radii. Determine the angle between them if the centre of gravity of the whole wire is at the centre of the arc.

$$\tan^{-1}\frac{4}{3}.$$

9. Determine the centre of gravity of a loop of the curve $r = a \cos 2\theta$.

$$\bar{x} = \frac{128\sqrt{2}}{105\pi} a.$$

10. Determine the centre of gravity of that part of the area of the cardioid in Fig. 61 which is on the right of the axis of $y$.

$$\bar{x} = \frac{16 + 5\pi}{16 + 6\pi} a; \; \bar{y} = \frac{10a}{8 + 3\pi}.$$

11. Find the centre of gravity of the arc of the same half cardioid.

$$\bar{x} = \bar{y} = \frac{4}{5} a.$$

12. Find the centre of gravity of the area above the axis of $x$ contained between the curves $y^2 = ax$ and $y^2 = 2ax - x^2$.

$$\bar{x} = a\frac{15\pi - 44}{15\pi - 40}; \; \bar{y} = \frac{a}{3\pi - 8}.$$

13. From a circle whose radius is $a$ two segments are cut off by chords drawn from the same point, each subtending $90^\circ$ at the centre. Find the distance from the centre of the centre of gravity of the remaining area.

$$\frac{2a}{3(\pi + 2)}.$$

14. The middle points of two adjacent sides of a square are joined, and the triangle formed by this straight line and the edges is cut off. Find the distance of the centre of gravity of the remainder from the centre of the square. $\frac{1}{21}$ of the diagonal.

15. If the small triangle in Ex. 14 is folded over instead of cut off, find the distance. $\frac{1}{48}$ of the diagonal.

16. Find the centre of gravity of the arc of the cycloid

$$x = a(\psi - \sin \psi), \qquad y = a(1 - \cos \psi).$$

$$\bar{x} = \pi a, \quad \bar{y} = \frac{4}{3} a.$$

17. Find the distance from the base of the centre of gravity of the area of the cycloid in Ex. 16.

$$\bar{y} = \frac{5}{6} a.$$

18. Find the centroid of the arc in the first quadrant of the four-cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$\bar{x} = \bar{y} = \frac{2}{5} a.$$

19. Find the centre of gravity of the area enclosed between the arc of Ex. 18 and the coordinate axes.

$$\bar{x} = \bar{y} = \frac{256a}{315\pi}.$$

20. Find the centre of gravity of the arc of the catenary, Fig. 48, Art. 132.

$$\bar{x} = x - c\frac{e^{\frac{x}{c}} + e^{-\frac{x}{c}} - 2}{e^{\frac{x}{c}} - e^{-\frac{x}{c}}}; \; \bar{y} = \frac{cx}{2s} + \frac{y}{2}.$$

21. An area is composed of a semi-ellipse and a semi-circle having the minor axis for its diameter. Find the distance of the centre of gravity from the common centre.

$$\frac{4(a - b)}{3\pi}.$$

22. Prove that the area between the parabolas $y^2 = px$ and $x^2 = qy$ is $\frac{1}{3}pq$, and find the coordinates of its centroid.

$$\bar{x} = \frac{9}{20}p^{\frac{1}{3}}q^{\frac{2}{3}}; \; \bar{y} = \frac{9}{20}p^{\frac{2}{3}}q^{\frac{1}{3}}.$$

23. Find the volume of a cone by Pappus's Theorem.

24. Show that the outer part of the anchor-ring (generated by the semi-circle $DNC$, Fig. 62) exceeds the inner part by twice the volume of the sphere whose radius is $a$.

25. An ellipse whose semi-axes are $a$ and $b$ revolves about a tangent at the extremity of the major axis. Find the volume generated. $2\pi^2 a^2 b$.

26. Find the volume enclosed between the surface of the solid of Ex. 25 and a tangent plane. $\frac{10 - 3\pi}{6}\pi a^2 b$.

## X.

### The Centre of Gravity of Particles not in One Plane.

**176.** We have seen in Art. 156 that the point through which the resultant of the weights of particles in a single plane passes when the plane is horizontal, is such that the resultant passes through it when the plane is inclined in any way; in other words, when gravity has any direction with respect to the plane. This of course applies to a heavy lamina, and is the basis of an experimental method of determining the centre of gravity. For, if the lamina be suspended from any point, the centre of gravity will, in equilibrium, be vertically beneath the point of suspension, because the resultant weight and the supporting force must have the same line of action. This will enable us to draw a line on the lamina upon which the centre of gravity must lie. By suspending the lamina from a point not in this line, we can determine another line containing the centre of gravity; this point must therefore be the intersection of the two lines.

If the lamina were suspended from a third point, the vertical through the point of suspension would be found to pass through the point so found, thus giving an experimental verification of the existence of a centre of gravity which is independent of the direction of gravity relatively to the body.

We have now to show that such a point, independent of the direction of gravity, exists for particles not all in one plane, and hence also for any solid body. This point is the centre of gravity, and, if the body were suspended from any point, would always be found in the vertical through the point of suspension.

177. In the case of particles, this point may be found by an extension of the graphic process in Art. 141, which it will be noticed is independent of the direction of gravity. For example, if there be a fourth particle at a point $D$, not in the plane of $ABC$, Fig. 50, and if $l$, $m$, $n$, $p$ be the weights of the particles, the centre of gravity $G$ of the four particles will be on the line $OD$, and will divide it inversely in the ratio of the weight of the particle $p$ at $D$ to the combined weights of the other three particles at $O$. That is to say,

$$OG : GD = p : l + m + n.$$

The same point would be found by grouping the particles in any other way. In fact, the construction shows that $G$ is situated in the plane passing through two of the particles and the centre of gravity of the other two. The six planes of this character meet in a point, and any three of them which pass through edges of the tetrahedron $ABCD$ not meeting in a point would serve to determine the centre of gravity.

In like manner, for any number of particles, the centre of gravity may be found, as in Art. 151, by substituting at each step for two of the particles a particle equal to their sum at their centre of gravity, the process applying as well when the particles are not as when they are in one plane. In particular, when the particles are equal, the centre of gravity is the centre of position of the points (see Art. 69), of which it was shown that the distance from any plane is the average of the distances of the particles from that plane.

## Statical Moments with Respect to the Coordinate Planes.

178. In the analytical treatment of the centre of gravity, the positions of the particles are referred to three rectangular axes, and we shall suppose at present that the plane of $xy$ is horizontal. Let the particles whose weights are $P_1, P_2, \ldots P_n$ be situated at the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $\ldots$ $(x_n, y_n, z_n)$. Since the weights of the particles act in lines parallel to the axis of $z$, their resultant is (by the principle of the transmission of force) the same

as if they were situated at their projections in the plane of $xy$; that is to say, at the points $(x_1, y_1)$, $(x_2, y_2)$, ... $(x_n, y_n)$. It follows that the resultant weight of the system of particles acts in a line which pierces the plane of $xy$ at the point $(\bar{x}\ \bar{y})$, defined by the equations of Art. 152; that is,

$$\bar{x}\Sigma P = \Sigma Px, \qquad \bar{y}\Sigma P = \Sigma Py.$$

The terms $P_1x_1$, $P_2x_2$, etc., which make up $\Sigma Px$, are now the products of the particles, each multiplied by its distance from the plane of $yz$. Such a product is called *the statical moment of the particle with respect to the plane.* Accordingly, $\Sigma Px$ is called the total or resultant statical moment of the system of particles with respect to the plane of $yz$.

**179.** In considering the various positions of a body, or of a system of particles supposed to be rigidly connected so that they maintain their relative positions, it is convenient to retain the coordinate axes in their positions relative to the system, and to imagine the relative direction of the force of gravity to be changed. If then we suppose gravity to act in the direction of the axis of $x$, the resultant of the weights of the particles may be shown in like manner to act in a line piercing the plane of $yz$ at the point $(\bar{y}, \bar{z})$ defined by the equations

$$\bar{y}\Sigma P = \Sigma Py, \qquad \bar{z}\Sigma P = \Sigma Pz.$$

The value of $\bar{y}$ is the same as before; and the lines of action in the two cases intersect in the point $(\bar{x}, \bar{y}, \bar{z})$, all of whose coordinates are defined by means of the statical moments of the particles* with respect to the coordinate planes. These coordi-

* We do not speak of the moment of a *force* or of a *system of forces*, in general, with respect to a plane; but the notion of the statical moment of a particle with respect to a plane arises from that of the moment of a force about an axis. The forces involved in the idea of the statical moment of a system of particles with respect to a plane are parallel forces acting in some direction parallel to the plane, and the axis is any line in the plane perpendicular to that direction.

nates are *the weighted means* of the like coordinates of the particles.

**180.** To show that the resultant of the weights passes through $(\bar{x}, \bar{y}, \bar{z})$ for all directions of the force of gravity relative to the coordinate planes, let $l$, $m$ and $n$ be the direction cosines (see Art. 66) of the direction of gravity, and resolve each of the forces into components parallel to the three axes. Then $X_1 = lP_1$, $Y_1 = mP_1$, $Z_1 = nP_1$, $X_2 = lP_2$, etc., and

$$\Sigma X = l\Sigma P, \qquad \Sigma Y = m\Sigma P, \qquad \Sigma Z = n\Sigma P.$$

The system of forces is now resolved into three systems of parallel forces. The resultant of the $Z$-system acts in a line parallel to the axis of $z$ and piercing the plane of $xy$ in the point $(\bar{x}, \bar{y})$; for we have seen that the resultant of the original system acts in this line when the $P$'s have this direction, and the $Z$-system consists of the same forces each multiplied by the constant factor $n$. In like manner the resultant of the $Y$-system is $m\Sigma P$, acting in a line parallel to the axis of $y$, which pierces the plane of $xz$ in the point $(\bar{x}, \bar{z})$, and that of the $X$-system is $l\Sigma P$, acting parallel to the axis of $x$ at the point $(\bar{y}, \bar{z})$ in the plane of $yz$. Thus the whole system is reduced to three forces acting in lines which meet in the point $(\bar{x}, \bar{y}, \bar{z})$. Therefore the resultant is the force $\Sigma P$ acting, as was to be proved, in a line which always passes through this point, which is for that reason called the centre of gravity.

### Centre of Gravity of a Volume or a Homogeneous Solid.

**181.** The position of the centre of gravity of a solid depends not only upon its size and shape, but upon the distribution of the matter within the volume. When the weights of equal volumes taken from any part whatever of the given volume are equal, the body is said to be *homogeneous*, and the weight of a unit volume is taken as the measure of *the density*. Denoting this weight by $w$, we have, for the weight of the homogeneous body of volume $V$, $W = wV$.

The distance of the centre of gravity from any plane is now found by the condition that the statical moment of $W$ at that point, with respect to the plane, is equal to the total moment of the elements. The weight of the element of volume, $dV$, is $wdV$; hence we have, for the moment with respect to the plane of $yz$,

$$\bar{x}W = \int wxdV,$$

or, dividing by $w$, since $W = wV$,

$$\bar{x}V = \int xdV. \quad \ldots \ldots \ldots \quad (1)$$

The value of each member of this equation is called the statical moment of the volume.

**182.** For example, let it be required to find the centre of gravity of a right cone of radius $b$ and height $a$. Fig. 64 represents a section through the geometrical axis upon which the centre of gravity obviously lies. Let this axis be taken as the axis of $x$, and the vertex $O$, as the origin. The section made by a plane perpendicular to the axis at the distance $x$ from the vertex is $\pi y^2$, and from Fig. 64 we have

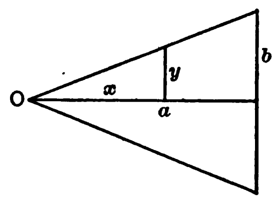

FIG. 64.

$$y = \frac{bx}{a}; \qquad \text{whence} \qquad dV = \frac{\pi b^2 x^2}{a^2} dx;$$

and equation (1) gives

$$\bar{x}V = \frac{\pi b^2}{a^2}\int_0^a x^3 dx = \frac{\pi b^2 a^2}{4}.$$

Hence, knowing the volume of the cone to be $V = \frac{1}{3}\pi b^2 a$, we have $\bar{x} = \frac{3}{4}a$; that is, the centre of gravity of a cone is at a distance of one-fourth of the altitude from the centre of the base.

## Employment of Triple Integration.

**183.** In the preceding article we supposed the area of the element for which the arm of the moment is constant to be known, and also the volume of the body. But the complete problem of finding the moment when only the equations of the bounding surfaces are known is, like that of finding the volume itself, one of triple integration. Suppose, in the first place, that the three independent variables used are the rectangular coordinates $x$, $y$ and $z$; then the ultimate element of volume is $dx\,dy\,dz$, and that of the moment with respect to the plane of $yz$ is $x\,dx\,dy\,dz$. Hence the moment is the triple integral of this expression taken with the same limits that would be used in finding the volume by the triple integration of $dx\,dy\,dz$.

**184.** For example, let us determine the centre of gravity of the solid represented in Fig. 65, which is common to the cylinder

$$z^2 + y^2 = a^2,$$

and the half, on the right of the plane of $yz$, of the cylinder

$$x^2 + y^2 = a^2.$$

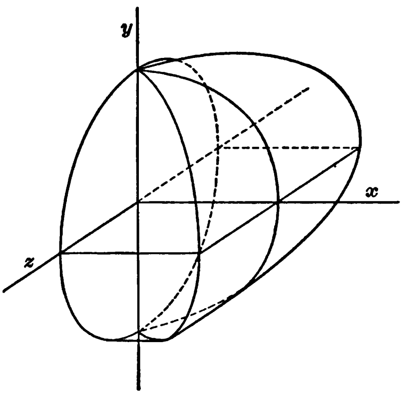

FIG. 65.

By symmetry, this centre of gravity is on the axis of $x$. The element of moment with respect to the plane of $yz$ is $x\,dx\,dy\,dz$; and, if the integrations are performed in the order $x$, $z$, $y$, we have

$$\bar{x}V = \int_{-a}^{a}\int_{-z_1}^{z_1}\int_{0}^{x_1} x\,dx\,dz\,dy,$$

in which $x_1$ and $z_1$ are limiting values determined by the equations of the bounding surfaces. The fact that $x$ occurs only in the equation of the second cylinder shows that the whole volume can be covered by one integration, in which the limit $x_1$ is taken from that equation.* Hence, performing the integration for $x$, and substituting the value of $x_1$, we have

$$\bar{x}V = \tfrac{1}{2}\int_{-a}^{a}\int_{-z_1}^{z_1}(a^2 - y^2)dz\,dy. \quad . \quad . \quad . \quad . \quad (2)$$

Next, performing the $z$-integration,

$$\bar{x}V = \int_{-a}^{a}(a^2 - y^2)^{\frac{3}{2}}dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Finally, putting $y = a\sin\theta$ in this integral,

$$\bar{x}V = a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta = 2a^4\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{3}{8}\pi a^4. \quad . \quad . \quad (4)$$

(Int. Calc., formula (P), p. 120.)

Following the same order of integration, we have, for the value of $V$,

$$V = \int_{-a}^{a}\int_{-z_1}^{z_1}\int_{0}^{x_1}dx\,dz\,dy = \int_{-a}^{a}\int_{-z_1}^{z_1}(a^2 - y^2)^{\frac{1}{2}}dz\,dy$$

$$= 2\int_{-a}^{a}(a^2 - y^2)dy = \frac{8a^3}{3}.$$

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* If this were not the case, it would be necessary to find the volume in two parts. For instance, in the present example, if the $y$-integration were performed first, the limiting value of $y$ would in a part of the volume be determined by one of the cylinders, and in the other part by the other.

Substituting in the value found for $\bar{x}V$,* we find

$$\bar{x} = \frac{9\pi a}{64}.$$

## Solids of Variable Density.

**185.** For a solid which is not homogeneous, let the variable $w$ denote the density, at any point; that is to say, the weight † of a homogeneous unit of volume having the density of the body at that point. If now the law of the distribution of the matter within a solid of given volume is known—in other words, if $w$ is given in the form of a function of the coordinates of the point—the weight of the body, as well as its statical moments, will have to be found

* In finding a volume by triple integration, the first two integrations are equivalent to finding the area of a section parallel to one of the coordinate planes; and if we can employ a single integral, it is because this area is already known. So also in the example of Art. 182, in finding the value of $\int\int\int x\,dx\,dy\,dz$, we were able to begin with the form $\int x\,dV$, because the result of the first two integrations would be the area of the section perpendicular to the axis of $x$, which in that case was known to be $\pi y^2$, $y$ being a given function of $x$. But, in the example of Art. 184, it was not convenient to find the area of a section parallel to the plane of $yz$.

We might, however, have used the section parallel to the plane of $xz$, which is the double square $2x^2$, so that the element is $2x^2dy$; because we know that its centre of gravity is at its geometric centre, and therefore at a distance $\frac{1}{2}x$ from the plane of $yz$. Therefore the element of moment is $x^3dy$, giving at once the above expression for $\bar{x}V$ as a simple integral, equation (3).

† The density is often defined as the *mass* of a unit of volume, so that the weight of the unit is $w = g\rho$, but in using gravitation units it is more convenient to use the weight of a unit volume, which is here denoted by $w$ and called density. This measure of density would be properly called specific weight had not the term *specific gravity* been applied to the *ratio* of density to that of water, for which $w$ is $62\frac{1}{2}$ pounds.

by integration. In this case, $dV$ being an element of volume, $wdV$ will be the element of weight, and the total weight will be

$$W = \int wdV$$

taken between the limits which define the volume.

**186.** For example, let us determine the weight of a sphere whose radius is $a$, when the density varies inversely as the square of the distance from the centre, and is $w_1$ at the surface. The conditions give $w : w_1 = a^2 : r^2$, whence $w = \frac{w_1 a^2}{r^2}$, where $r$ is the distance of the point from the centre. Since $\rho$ is in this case given in terms of $r$, it is convenient to use an element of volume such that $r$ has the same value for all its points. The area of the spherical surface at the distance $r$ is $4\pi r^2$; hence, taking as the element the spherical shell of thickness $dr$, we have $dV = 4\pi r^2 dr$. Therefore

$$W = \int wdV = 4\pi w_1 a^2 \int_0^a dr = 4\pi w_1 a^3.$$

Since the volume of the sphere is $\frac{4}{3}\pi a^3$, this sphere has three times the weight of a homogeneous sphere of density equal to that at the surface. Hence its *average density* is $3w$.

## Centre of Gravity of a Solid of Variable Density.

**187.** In finding the statical moment of a solid which is not homogeneous, it will generally be necessary to use the ultimate element of volume as in Art. 184, because the lamina parallel to the plane of reference, used as an element in Art. 182, will not have the same value of $w$ for all points of its area; and therefore, although we may know its area, we cannot write the expression for its weight. For example, let the centre of gravity of one half of the sphere considered in the preceding article be required.

The centre of gravity is in the radius perpendicular to the base of the hemisphere. Let $BAC$, Fig. 66, be a section through this radius $OA$, intersecting the base in the diameter $BC$, and $P$ any point within the hemisphere. The fact that $w$ is a function of $r$ makes it advisable to take $r$ as one of the three independent variables; for the other two, let us take $\theta$ the angle $BOP$, and $\phi$ the angle between the planes $BOP$ and $BOA$. When these quantities vary separately, the differentials of the motion of $P$ are $dr$, $rd\theta$ and $r \sin \theta d\phi$; and, since these differentials are mutually rectangular, the element of volume is $r^2 \sin \theta \, d\theta \, d\phi \, dr$, and that of weight is

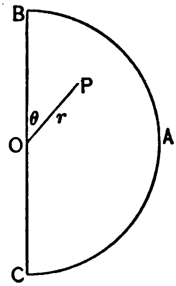


FIG. 66.

$$dW = r^2 w \sin \theta \, d\theta \, d\phi \, dr.$$

The distance of $P$ from the base of the hemisphere or arm of the moment, is

$$r \sin \theta \cos \phi;$$

hence the element of moment is $r^3 w \sin^2 \theta \cos \phi \, d\theta \, d\phi \, dr$, and we may write

$$\bar{x} W = \int_0^a wr^3 dr \, . \int_0^\pi \sin^2\theta d\theta \, . \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos \phi d\phi,$$

which is the product of three simple integrals, because the limits of integration are independent. The value of the second integral is $\frac{1}{2}\pi$, and that of the third is 2; hence, substituting $w = \dfrac{w_1 a^2}{r^2}$, we have

$$\bar{x} W = w_1 \pi a^2 \int_0^a r dr = \frac{w_1 \pi a^4}{2};$$

and since the weight of the hemisphere as found in Art. 186 is $W = 2\pi w_1 a^3$, we derive $\bar{x} = \frac{1}{4}a$.

## Stable and Unstable Equilibrium.

**188.** We have seen that, when the conditions of a problem define the forces which act upon a body for all positions of the body, or at least for a series of positions which the body is free to take, there are positions of equilibrium. If now the body in a position of equilibrium suffer any of its possible displacements, the lines of action, and sometimes also the magnitudes, of the forces will be so modified that equilibrium will, in general, no longer exist. If the action of the forces in the new position, assumed to be indefinitely near to that of equilibrium, is such as to cause the body to return to the position of equilibrium, that position is said to be one of *stable equilibrium.* If, on the other hand, the action is such as to urge the body away from the position of equilibrium, it is said to be one of *unstable equilibrium.*

**189.** For example, we supposed the weights in Fig. 11, page 38, to be allowed to adjust themselves into a position of equilibrium. This would not be possible if it were not a position of stable equilibrium, but a little consideration will show that this is the case. For instance, if $C$ be displaced downward, the resultant of $P$ and $Q$ will become greater than $R$, and the total action on the knot will be upward.

Again, in Fig. 21, page 54, the position is one of stable equilibrium, because, if $A$ be brought nearer to $C$, the repulsive force is increased.

In Fig. 20, page 51, $P$ was so determined as to produce equilibrium, and we cannot pronounce it as stable or unstable unless $P$ is defined for all positions of $A$ regarded as movable along the line $AB$. If now we suppose $P$ to remain constant in magnitude and direction, equilibrium will still exist when $A$ is displaced. The position is therefore called one of *neutral* or *astatic* equilibrium.

**190.** When a rigid body is displaced in any manner involving rotation, the forces of gravity upon the several parts retain their directions and magnitudes, while their lines of action are shifted into new relative positions. In proving the existence of a centre

of gravity, we have shown that there is a point at which if the body be supported it will remain in equilibrium for all possible displacements, that is, it will be in astatic equilibrium. Accordingly a system of parallel forces having definite points of application in a body is said to have an *astatic centre.*

In the case of a system of coplanar, but not parallel, forces having definite points of application in a rigid body, and invariable in direction and magnitude when the body is turned in the plane, it can also be shown that an astatic centre exists. See examples 23 and 24 below.

**191.** If the heavy rigid body be supported at any other point than the centre of gravity, the reaction of the support will be equal to the weight, and with it will form a couple which will cause the body to turn, if free to do so, unless the centres of gravity and of suspension are in a vertical line. In the latter case, equilibrium will exist, and it will plainly be unstable when the centre of gravity is above the point of support, and stable when it is below it.

We may, in this case, regard the centre of gravity as a heavy particle which is constrained to lie in a spherical surface, and therefore rests in stable equilibrium only at the lowest point of the surface. Again, if the body be supported upon an axis about which it is free to turn, the centre of gravity describes a circle (unless the axis passes through it), and will seek the lowest point of the circle if it lies in a vertical or oblique plane; but, if the plane of the circle is horizontal, that is, if the axis is vertical, the body will be in neutral equilibrium.

## Equilibrium in Rolling Motion.

**192.** When a body with a curved surface rolls upon a fixed surface, equilibrium can exist only when the point of contact is in a vertical line with the centre of gravity; otherwise there will be, as in the preceding article, a couple which will cause the body to roll.

In some cases, the stability of the equilibrium is readily deter-

mined by considering the path of the centre of gravity. For example, when a cylinder rolls on a horizontal plane, if the centre of gravity is not on the geometrical axis it will obviously be at the lowest point of its path when between the axis and line of contact, and at the highest point when vertically above the axis. The former is therefore a case of stable, and the latter one of unstable, equilibrium.

**193.** In general, the stability of the equilibrium is more conveniently determined by means of the couple formed when displacement takes place. For example, suppose the heavy body to rest with its convex surface in contact with the convex surface of a fixed body, the common tangent plane being horizontal and the centre of gravity $G$ vertically above the point of contact, so that equilibrium exists. Let Fig. 67 represent a vertical section through these points, and let the sections of the surfaces at first be supposed circles whose centres are $C$ and $B$ and whose radii are $r$ and $R$. If the surfaces are smooth, the equilibrium is unstable, because as soon as displacement takes place the body will slide down the inclined surface. But suppose them to be rough, so that rolling takes place, and let $C'a$ be the new position of the radius $CA$ upon which $G$ lies, while $A'$ is the new point of contact. Then the arcs $A'a$ and $AA'$ are equal; and, denoting the angles subtended at $C'$ and $B$ by $\phi$ and $\theta$, we have

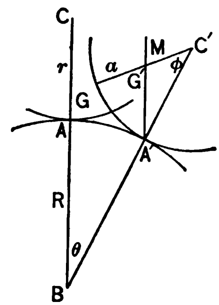


Fig. 67.

$$r\phi = R\theta, \qquad \text{or} \qquad \phi : \theta = R : r. \quad . \quad . \quad . \quad (1)$$

Let a vertical line through $A'$ intersect $C'a$ in $M$; then, if $G'$ is between $C'$ and $M$, the couple formed by the vertical forces, namely, the weight acting at $G'$ and the upward reaction of the fixed surface at $A'$, will tend to roll the body still further from its original position, and the equilibrium is unstable. If, on the other hand, $G'$ is between $a$ and $M$, the body will tend to return

and the equilibrium is stable. Now, from the triangle $C'MA'$ we find

$$\frac{C'M}{r} = \frac{\sin\theta}{\sin(\theta+\phi)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The limiting value, when $\theta$ is small, of $\dfrac{\sin\theta}{\sin(\theta+\phi)}$ is $\dfrac{\theta}{\theta+\phi}$, or, by equation (1), $\dfrac{r}{r+R}$; hence at the limit

$$C'M = \frac{r^2}{r+R}, \qquad \text{and} \qquad aM = \frac{rR}{r+R}. \quad . \quad . \quad (3)$$

It follows that, if $AG$, the height of the centre of gravity above the point of contact in the position of equilibrium, is greater than $\dfrac{rR}{r+R}$, the position is one of unstable equilibrium, but if it is less than $\dfrac{rR}{r+R}$, the equilibrium is stable.

**194.** When the sections of the surfaces are not circles, the condition for stability is the same, $R$ and $r$ now standing for the radii of curvature of the sections. If the body rests in a concavity of the fixed body, $R$ is negative, and putting $R = -R'$, we have, for the value which $AG$ must not exceed if the equilibrium is to be stable,

$$\frac{rR'}{R'-r}.$$

In like manner, if the curvature of the section of the moving body be reversed, putting $r = -r'$, the expression for the limiting height becomes

$$\frac{r'R}{r'-R}.$$

If the body rests upon a plane, $R$ is infinite, and we have $r$ for the limiting value, as obviously should be expected. Again, if

$r$ is infinite, so that a plane surface rests upon a curved one, we have $R$ for the limiting value.

If the curvature of the sections made by vertical planes passing through the line $BC$ is variable, it is necessary for complete stability that $AG$ should be less than the least value of $\dfrac{rR}{r+R}$.

## Limits of Stability.

**195.** In cases of stable equilibrium, if the displacement be carried beyond certain limits, the body will not return to its original position. For example, in Fig. 67, though for small displacements $G$ is found on the left of the vertical through $A'$, it will, if the angle of rolling be increased, reach that line, and the body will then be in a position of unstable equilibrium. Hence if it be still further displaced, it will not tend to return to its first position. In like manner, there is a position of unstable equilibrium on the other side, and these determine an interval within which displacements may take place without causing the body to leave the position of stable equilibrium. The equilibrium is said to be more or less stable according to the size of this interval. In the example, this interval, which is large when $AG$ is small, decreases as we increase $AG$; and finally disappears when $AG$ equals the limiting value given in equation (3), so that the position then becomes one of unstable equilibrium.

**196.** The notion of limits of stability is sometimes applied also to cases in which the body itself is not displaced with reference to other bodies which are in contact with it and react upon it, but in which external forces can undergo changes within certain limits before equilibrium is destroyed.

For example, a three-legged table stands upon a horizontal plane. If the centre of gravity be moved, by changing the position of heavy bodies upon the table, the resistances at the three feet adapt themselves as explained in Art. 148. But, if it be moved until its projection upon the horizontal plane crosses one of the sides of the triangle formed by the feet, the equilibrium will no

longer exist, unless the resistance at the opposite foot can change sign; that is to say, unless this foot is held down, the table will topple over. Thus, the condition of equilibrium, when the feet are not held down, is that the projection of the centre of gravity shall fall within this triangle, which is called *the base*.

In like manner, for a body of any form resting upon a horizontal plane, the smallest convex polygon which encloses all the points of contact with the plane is called the base, and the condition of stability is that a perpendicular from the centre of gravity shall fall within the base.

EXAMPLES. X.

1. Determine the centre of gravity of seven equal particles situated at the vertices of a cube.

2. Show that the centre of gravity of a tetrahedron is the same as that of four equal particles at its vertices, and cuts off one-fourth of the line joining the centre of gravity of either face with the opposite vertex.

3. Extend the result of Ex. 2 to any pyramid and thence to any cone.

4. Show that if $a$ and $b$ be any homologous lines in the bases of a frustum, and $h$ the distance between the bases, the distance of the centre of gravity from the base in which $a$ lies is

$$\frac{a^2 + 2ab + 3b^2}{4(a^2 + ab + b^2)}h.$$

5. A cone of height $h$ is cut out of a cylinder of the same base and height. Find the distance of the centre of gravity of the remainder from the vertex. $\frac{3}{8}h.$

6. Find the centre of gravity of the solid formed by the revolution of the sector of a circle about one of its extreme radii.

The height of the cone being denoted by $h$, and the radius of the circle by $a$, we have $\bar{x} = \frac{3}{8}(a + h)$.

7. A solid is formed of a hemisphere whose radius is $a$ and a paraboloid with the same base. What must be the height of the

paraboloid, in order that the solid may rest with any point of the spherical surface upon a horizontal plane? $a\frac{\sqrt{6}}{2}$.

8. Find the distance between the centre and the centre of gravity of one-half an anchor-ring generated by a circle whose radius is $a$ and whose centre describes the circle whose radius is $b$.

$$\frac{4b^2 + a^2}{2\pi b}.$$

9. A paraboloid whose parameter is $4a$ stands on a plane whose inclination is $\alpha$ and is prevented from sliding. Find its height if just on the point of toppling over. $h = 36a \cot^2 \alpha$.

10. A paraboloid and a cone have a common base and vertices at the same point. Find the centre of gravity of the solid enclosed between their surfaces.

The centre of gravity is the middle point of the axis.

11. A cone of height $h$ and radius $a$ is hung up by a string over a smooth peg, one end being attached to the vertex and the other to the rim. Find the length of the string if equilibrium exists when the axis is horizontal. $\sqrt{(4a^2 + h^2)}$.

12. Determine the centre of gravity of the surface formed by revolving the cardioid $r = a(1 + \cos \theta)$ about the initial line.

$$\bar{x} = \frac{50}{63}a.$$

13. A frustum is cut from a right cone by a plane bisecting the axis. If the frustum rests in equilibrium with its slant height upon a horizontal plane, find the greatest ratio the radius of the lower base can bear to the height. $\sqrt{\frac{17}{7}}$.

14. Find the average density of a sphere whose density varies inversely as the distance from the centre, $\mu$ being the density at the surface. $\frac{3}{2}\mu$.

15. The corners of a tetrahedron are cut off by planes parallel to the opposite faces. Prove that if the parts cut off are equal, the centre of gravity of the remainder will coincide with that of the tetrahedron.

16. If a uniform lamina, whose form is that of the area between the sinusoid $y = \sin x$ and the axis of $x$, be suspended from one extremity of its base, show that the base will make the angle $\tan^{-1} 4$ with the horizontal.

17. The density of a sphere of radius $a$ varies uniformly from $\rho_0$ at the centre to $\rho_1$ at the surface. Determine the centre of gravity of one hemisphere.
$$\bar{x} = \frac{3(\rho_0 + 4\rho_1)a}{10(\rho_0 + 3\rho_1)}.$$

18. A cone of height $h$ rests with its base upon the vertex of a paraboloid whose parameter is $4a$. Find the greatest value of $h$ for stable equilibrium. $8a$.

19. A plank rests upon a rough cylinder of radius $R$ in a horizontal position of stable equilibrium, $h$ being the height of the centre of gravity above the point of contact. Show that the position of unstable equilibrium occurs when the plank is rolled through the angle $\theta$ determined by

$$\theta \cot \theta = \frac{h}{R}.$$

20. A paraboloid, whose height is $h$, and the radius of whose base is $b$, rests with its convex surface on a horizontal plane. Determine the inclination $\alpha$ of the axis to the horizon, and thence determine the smallest value of $h$ for which the equilibrium at the vertex is stable.
$$\sin^2 \alpha = \frac{3b^2}{8h^2 - 3b^2}.$$

21. Find the distance between the centre of the sphere and the centre of gravity of the volume cut from the sphere of Art. 186 by a cone whose vertex is at the centre and whose semi-vertical angle is $\alpha$.
$$\frac{a}{4}(1 + \cos \alpha).$$

22. Determine the centre of gravity of a segment of the same sphere cut off by the plane $x = h$, knowing that the centre of gravity of a spherical cap bisects its altitude.
$$\bar{x} = \frac{1}{4}\frac{a^2 - h^2 - 2h^2(\log a - \log h)}{a - h - h(\log a - \log h)}.$$

23. If a rigid body, upon which two coplanar forces act at definite points of application, be turned in the plane of the forces, the forces retaining their magnitudes and directions, show that the resultant will always pass through a fixed point upon the circumference which is the locus of the intersection of the lines of action.

This point is the *astatic centre* of the forces (see Art. 190), and when they are parallel it becomes the "centre of parallel forces."

24. Show, hence, that any system of coplanar forces has an astatic centre ; and, if the forces are referred to rectangular axes, putting

$$\Sigma(xY - yX) = K \quad \text{and} \quad \Sigma(xX + yY) = V,$$

prove that the astatic centre is the intersection of the lines

$$x\Sigma Y - y\Sigma X = K \quad \text{and} \quad x\Sigma X + y\Sigma Y = V.$$

# CHAPTER V.

## FRICTIONAL RESISTANCE.

### XI.

### Laws of Friction.

197. When a body is so constrained, by material bodies with which it is in contact, that motion can take place only along a certain line, the resistance of the line is normal to it when the surfaces in contact are smooth; but when they are *rough*, the line offers a resistance which has a component along as well as one normal to it. The component of resistance along the line is called *statical friction.*

This frictional resistance, like the normal resistance, is a passive force which adapts itself so as to produce equilibrium if possible; but, unlike the normal resistance, it cannot exceed a certain limit. Thus, if a brick rest upon a horizontal table, and a small horizontal force applied to it be gradually increased, this force will be resisted until it reaches a certain value which is called *the limiting statical friction.* If the force exceed this value, the brick will move, but with the acceleration due to a force less than that actually applied, and the diminution thus suffered by the force is called *the dynamical friction.*

198. The following "laws of friction" were enunciated in 1781 by Coulomb as the results of his experiments:

1. *The limiting statical friction is, for a given pair of surfaces in contact, proportional to the normal pressure.* Thus, if a second brick of the same weight be placed upon the brick in the illustration above it is found that the limiting statical friction is doubled.

2. *The limiting statical friction is independent of the area of surface in contact.* Thus, if the brick be placed upon its side, the horizontal force required to move it is the same as when it rests upon its face. This law is easily seen to be a consequence of the first law. For, if the second brick be placed on the table and connected by a string to the first, the statical friction is doubled, and is therefore by the first law equal to the value which it has when the normal force is doubled, without change of the area of contact, by placing the second brick upon the first.

3. *The dynamical friction is independent of the velocity.* Accordingly, after the body is in motion with a given velocity, the same force will suffice to keep it in uniform motion, no matter what the given velocity may be; and, if the force applied exceed this, the acceleration will be constant.

We should, therefore, expect the dynamical friction to be equal to the limiting statical friction and to obey the same laws, but it is found that the dynamical friction is somewhat the smaller. The third law cannot therefore be true for very low velocities, and it has also been found that the laws require modification in certain other extreme cases. But we are here concerned only with the limiting statical friction, and shall assume that, in accordance with the first two laws, it bears a fixed ratio to the normal resistance $R$; so that it may be written $\mu R$, in which $\mu$ is called *the coefficient of friction.*

**199.** The cause of friction is the *roughness* of surfaces consisting of small projections which, fitting into one another, must either be broken off, or cause the surfaces to separate when they move on one another. The coefficient of friction differs greatly for different substances, and is diminished by grinding and polishing the surfaces, and also by introducing lubricating substances. The following table will give an idea of the general range of values of $\mu$ for unlubricated surfaces:

| | | | |
|---|---|---|---|
| For iron on stone, | $\mu$ varies from | .3 | to .7; |
| For timber on timber, | $\mu$ " | .2 | to .5; |
| For timber on metals, | $\mu$ " | .2 | to .6; |
| For metals on metals, | $\mu$ " | .15 | to .25. |

**200.** The direction of frictional resistance in the case of a body resting upon a surface is opposite that in which motion would take place if the surface were smooth. The resultant of the normal and the frictional resistances is called *the total resistance* of the surface. Thus, if a particle at $A$, Fig. 68, acted upon by forces, be kept at rest by the normal resistance $AR$ of the plane $AB$ and the friction $AF$, the total resistance of the plane is represented by $AT$, the resultant of $AR$ and $AF$. This force must, of course, be directly opposite to the resultant of the active forces which would otherwise produce motion. Hence, if $AB$ is an inclined plane and there is no active force except the weight of the particle, $AT$ will be directed vertically upward.

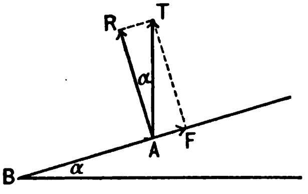


FIG. 68.

**201.** If we furthermore suppose the friction acting to be the maximum or limiting statical friction, we have, denoting the normal resistance by $R$ and the friction by $F$,

$$F = \mu R,$$

and, if we denote by $\alpha$ the angle between the normal and the total resistance when limiting friction is acting,

$$\tan \alpha = \frac{F}{R} = \mu.$$

This angle $\alpha$, which is therefore the greatest possible inclination of the total resistance to the normal, is called *the angle of friction.* Fig. 68 shows that the angle of friction is the same as the inclination of the plane when the body is about to slide. It may be found by gradually increasing the inclination until motion takes place;* the total resistance is, in this case, equal

* Owing to the fact that the limiting statical exceeds the dynamical friction, the body will, on starting, move with a uniform acceleration. If the inclination be now decreased until its tangent equals the coefficient of dynamical friction, the body will move with a uniform velocity.

to the weight $W$, hence the normal resistance produced by the weight is $W \cos \alpha$, and the friction is $W \sin \alpha$.

## Limits of Equilibrium on a Rough Inclined Plane.

**202.** Suppose the weight $W$ resting at $A$ upon a rough plane, whose inclination $\theta$ is greater than the angle of friction, to be supported by the force $P$, whose line of action lies in the vertical plane which contains the normal, and makes the angle $\phi$ with the inclined plane, as represented in Fig. 69; let us find the limiting values of $P$ consistent with equilibrium.

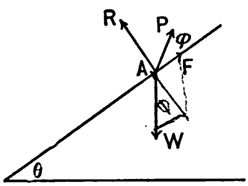


FIG. 69.

The greatest value of $P$ will occur when the body is on the point of moving up the plane, and the least value is that which is just sufficient to prevent the body sliding down the plane.

In the latter case, the friction acts up the plane, as represented in the figure (that is, it assists in holding the body up), and, being the limiting friction, its value is $\mu R$, where $R$ denotes the normal resistance. Resolving the forces perpendicularly to and along the plane, we have

$$R = W \cos \theta - P \sin \phi,$$

$$\mu R + P \cos \phi = W \sin \theta.$$

Eliminating $R$,

$$P(\cos \phi - \mu \sin \phi) = W(\sin \theta - \mu \cos \theta);$$

and, since $\mu = \tan \alpha$, where $\alpha$ is the angle of friction, we have for the least value of $P$, or *the force which just sustains the weight*,

$$P = W \frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\cos \phi \cos \alpha - \sin \phi \sin \alpha} = W \frac{\sin (\theta - \alpha)}{\cos (\phi + \alpha)}. \qquad (1)$$

**203.** To find the greatest value of $P$, the body being on the point of moving up the plane, we have only to change the direction of the frictional force $\mu R$ in the figure, since the limiting friction now acts down the plane. Hence, replacing $\mu$ by $-\mu$, or $\alpha$ by $-\alpha$, in equation (1), we have for the greatest value of $P$, or *the force which will just fail to move the body up the plane*,

$$P = W\frac{\sin(\theta + \alpha)}{\cos(\phi - \alpha)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equilibrium will exist for any value of $P$ between the limits given in equations (1) and (2).

**204.** When $\theta = \alpha$, $P$ in equation (1) vanishes, irrespective of the value of $\phi$, as should be expected, since friction alone will just sustain the body at this inclination of the plane.

When $\theta < \alpha$, $P$ in equation (1) becomes negative, and putting $P'$ for its numerical value, $P'$ is the force which, acting in a direction opposite to $AP$ in Fig. 69, will just fail to move the body down the plane. This force acts down the plane, and with a component pushing the body against the plane. Replacing $P$ by $-P'$ in equation (1), we have

$$P' = W\frac{\sin(\alpha - \theta)}{\cos(\alpha + \phi)}.$$

The angle between the direction of the force $P'$ and the direction down the plane now lies below the plane; hence, if we put $\phi' = -\phi$ in the last equation, we shall have the value of $P'$ when pulling down the plane at the inclination $\phi'$. That is to say, for *the greatest force which fails to move the body down the plane*, we have

$$P' = W\frac{\sin(\alpha - \theta)}{\cos(\alpha - \phi')}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This formula may of course be derived directly from a diagram properly constructed. In each case, the value of $P$ might have

been obtained without elimination by resolving in a direction perpendicular to that of the total resistance.

**205.** The values of the force in equations (2) and (3) may be regarded as the *least values of the force which will start the body* up or down the plane, as the case may be, when acting at the given inclination $\phi$. Hence, when $\phi$ is arbitrary, the value which makes the force a minimum is the most *advantageous* when the body is to be moved along the plane. Thus, the most advantageous value of $\phi$ for hauling the body up the plane is $\phi = \alpha$, which makes $P$ in equation (2) a minimum. As we increase the value of $\phi$ from zero, the component of $P$ along the plane which must overcome the friction (as well as a component of $W$) is diminished, but this loss is compensated by the diminution of friction produced by the considerable component of $P$ normal to the plane. The corresponding value of $P$ is $W \sin (\theta + \alpha)$, which is the same as if the plane were smooth and its inclination were $\theta + \alpha$.

The most advantageous value of $\phi$ for keeping the body from sliding down the plane when $\theta > \alpha$ is $\phi = -\alpha$, which makes $P$ in equation (1) a minimum. This implies that, if the force be applied from above the plane, it should be a pushing force up the plane, a component of which increases the friction, which is now advantageous. The corresponding value of $P$ is $W \sin (\theta - \alpha)$, the same that would be required on a smooth plane inclined at the angle $\theta - \alpha$.

Again, the most advantageous value of $\phi$ for hauling the body down the plane, when $\theta < \alpha$, is, from equation (3), $\phi' = \alpha$; thus, for hauling in either direction the best "angle of draught" is the angle of friction, the direction of draught being in each case perpendicular to the direction of total resistance.

## The Cone of Friction.

**206.** We have, in the foregoing articles, considered the force $P$ in Fig. 69 as acting in the plane containing the normal at $A$ and the vertical. When this restriction is removed, the total

resistance of the rough plane will not necessarily lie in the vertical plane. But equilibrium will exist whenever the angle which its direction makes with the normal does not exceed $\alpha$. The limiting positions of the total resistance will, therefore, lie in the surface of a cone of which $A$ is the vertex, the normal is the axis and $\alpha$ is the semi-vertical angle. This cone is called *the cone of friction.*

**207.** As an application to the general case of a body resting upon a rough inclined plane, we notice that $P$ is in equilibrium with the weight $W$ and the total resistance, which we shall denote by $T$. Therefore $P$ is the resultant of $W$ and $T$ both reversed. When the limiting total resistance is reversed, it acts in the surface of the cone of friction produced downward below the inclined plane. Hence, if $T$ were known, we could construct $P$ at $A$ by first laying off $W$ reversed (that is, upward), and then from its extremity a line representing $T$ parallel to an element of the cone just mentioned. It follows that, if we lift this cone, without any change of direction, until its vertex is at the extremity of $W$ laid off upward, the end of the line representing $P$ will lie in the surface of the cone.

In other words, let a right circular cone be constructed with its vertex at a distance $W$ directly above $A$, its axis perpendicular to the inclined plane, and its semi-vertical angle equal to $\alpha$. Then the body at $A$ will be in equilibrium when acted upon by a force $P$ represented by a line drawn from $A$ to any point within this cone; but, if the line representing $P$ terminates outside of the cone, the body will move.

The point $A$ will itself be outside of the cone when $\theta > \alpha$, as supposed in Art. 202, and within it when $\theta < \alpha$, as supposed in Art. 204.

## Frictional Equilibrium of a Rigid Body.

**208.** In the case of a body resting upon a curved surface or upon several surfaces, the friction at different points must be considered separately. When friction is called into action to

produce equilibrium, the resistances are, in general, indeterminate. Thus, if a heavy rod $AB$ rests with its ends upon a rough horizontal plane and a rough vertical wall, the total resistances at $A$ and $B$ will meet the vertical line through the centre of gravity in the same point. This point must be within each of the cones of friction constructed, as explained in Art. 206, at $A$ and $B$; otherwise, one at least of the frictional resistances would have to exceed its limiting value. Supposing the vertical line through the centre of gravity to pass through the space common to the two cones, the point in question may be *any* point of the segment of the line within this space, and the values of the resistances at $A$ and $B$ are to a certain extent indeterminate.

**209.** In the limiting position of equilibrium, however, we must assume that, at each of the points where motion must take place if the equilibrium is broken, the maximum friction in the direction opposite to that motion is called into action. For example, let us find the greatest angle $\theta$ which the rod $AB$ can make with the wall, supposing it situated in a vertical plane perpendicular to the wall, as represented in Fig. 70. The coefficients of friction at $A$ and at $B$ are assumed to be the same. In this case, if the equilibrium is broken, the motion which must take place at $A$ is outward from the wall, and that at $B$ is downward. Drawing the directions of the total resistances accordingly, so as to make the angle $\alpha$ with the normal in each case, they will meet in $C$, and the triangle $ABC$ is right-angled at $C$. Moreover, the point $C$ is vertically above the centre of gravity $G$; hence $ACG = \alpha$.

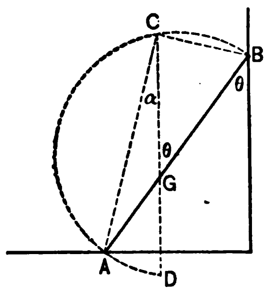

FIG. 70.

The circumference of a circle described on the diameter $AB$ will pass through $C$, and if $CG$ produced meets it in $D$, the arc $AD$ which subtends the angle $\alpha$ at $C$ will subtend the angle $2\alpha$ at the centre. Hence $D$ is a fixed point* relatively to $AB$. We

* The total resistances are $W \cos \alpha$ and $W \sin \alpha$, which are con-

have thus for a given position of $G$ a graphical construction for the angle $AGD$, which is $\theta$. If the rod is uniform, so that $G$ is at its middle point, we shall have $\theta = 2\alpha$.

## Moment of Friction.

**210.** When a definite portion of the surface of the solid body whose equilibrium is under consideration is in contact with the fixed body, the distribution of the normal pressure over the area of contact depends, as mentioned in Art. 158, upon the geometrical exactness of the surfaces in contact, and the rigidity of the materials.

In accordance with the laws of friction, Art. 198, this distribution of pressure makes no difference in the total amount of friction when the friction at different points acts in parallel lines; that is, when the motion resisted is one of translation. When, on the other hand, the motion resisted is one of rotation about an axis perpendicular to the surface of contact (which we shall suppose to be a plane area), this is no longer true. Thus, a heavy cylinder resting with its base upon a rough horizontal plane will, through friction, resist a force tending to turn it about its axis; but the maximum moment of this resistance depends upon the distribution of the pressure produced by the weight. If, owing to slight inaccuracies in the fitting of the surfaces, the weight rests chiefly upon the area near the centre of the base, the limiting moment of the friction will be small. If, on the other hand, it rests chiefly upon the rim, the moment will be comparatively large.

**211.** If, in the illustration of the cylinder, given in the preceding article, we assume the pressure caused by the weight $W$

stants; hence, if we suppose the limiting resistance to remain in action while the body is moved, $D$ will be their astatic centre (see Ex. X, 23). The forces are therefore equivalent to $W$ downward at $G$, and $W$ upward at $D$. This indicates unstable equilibrium. Practically the equilibrium on one side is neutral; for friction, being a passive force, cannot act so as to produce motion; in other words, when $\theta$ is diminished, the friction no longer has its limiting value.

to be uniformly distributed over the circular base whose radius is $a$, we shall have $p = \frac{W}{\pi a^2}$, while $pdA$ is the pressure upon an element of area $dA$, and $\mu pdA$ is the limiting friction. Hence, if $r$ be the distance of the element from the centre of rotation, $\mu prdA$ is the element of the moment of the friction about the axis of rotation, which we assume to be the geometrical axis of the cylinder. Taking, for the element of area, the ring which is at the distance $r$ from the centre, we have $dA = 2\pi r dr$; whence we find, for the element of moment,

$$dM = \frac{2\mu W}{a^2} r^2 dr,$$

and integrating,

$$M = \frac{2\mu W}{a^2}\int_0^a r^2 dr = \tfrac{2}{3}a\mu W.$$

Since the limiting friction is $\mu W$, the moment is the same as if the whole friction acted with an arm $\frac{2}{3}a$, so that it is $\frac{2}{3}$ of what it would be if the weight rested entirely upon the rim.

Supposing the axis to be fixed, a horizontal force whose moment does not exceed $\frac{2}{3}a\mu W$ can be applied to the cylinder without producing motion. Also, if the axis is not fixed, a horizontal couple not exceeding the same limit can be applied without producing motion, for it is easily seen that the moment of friction about any other point is greater than that about the centre.

**212.** The resistance of a body to rolling when in neutral equilibrium, as, for example, of a homogeneous cylinder upon a horizontal plane, is called *rolling friction*. Like true friction, its limiting value is proportional to the normal resistance, but its coefficient is, in general, very small, particularly when the substances in contact are hard.

### Friction of a Cord on a Rough Surface.

**213.** We have seen, in Art. 51, that the tension of a cord is not altered when the cord passes over a smooth curved surface,

because the resistance is always normal to the direction of the string. But, if the surface is rough, an inequality of tension may exist, the equilibrium being maintained by the friction of the cord upon the surface. In estimating the effect of this friction, it is necessary first to obtain an expression for the normal resistance of the surface at any point.

Suppose the curve of contact of the cord and surface to be the circular arc $AB$, Fig. 71. Denote the radius by $a$, the angle subtended at the centre by $\theta$, the length of arc by $s$, and the tensions at $A$ and $B$, which we shall at first suppose equal, by $T$.

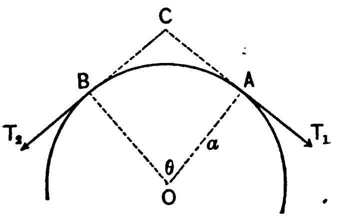


FIG. 71.

Producing the tangents at $A$ and $B$ to meet in $C$, we see that the resultant resistance of the whole arc $AB$ acts in the direction $CO$, which bisects the angle $AOB$, and its value is

$$2T \sin \tfrac{1}{2}\theta.$$

The tension being in this case uniform, it is obvious that the normal resistance is uniformly distributed over the arc. The intensity of the resistance at any point of the arc of contact is the resistance which would be offered by a unit's length, if at every point the resistance were the same in intensity and direction. Denoting this *resistance per linear unit* by $R$, the resistance offered by a length $s$ under the same circumstances would be $Rs$. It follows that the value of $R$ is the limit of the value of the resultant resistance divided by $s$ (that is, by $a\theta$), when $\theta$ is diminished without limit. We have, then,

$$R = \left.\frac{2T \sin \frac{1}{2}\theta}{a\theta}\right]_{\theta = 0} = \frac{T}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

**214.** Next suppose that the tensions $T_1$ at $A$ and $T_2$ at $B$ are not equal, and that $T_1 > T_2$. The component of the resistance in the direction $OC$ is now $(T_1 + T_2) \sin \frac{1}{2}\theta$. Proceeding to the

limit, we have for the normal resistance $R$ the same value as before, since $T_2 = T_1$ at the limit. But the resistance has now a component perpendicular to $OC$, namely,

$$(T_1 - T_2) \cos \tfrac{1}{2}\theta,$$

which balances the resolved parts of $T_1$ and $T_2$ in this direction. This is the resultant effect of the friction on the arc $AB$ or $a\theta$. It follows that the friction for any element of arc, $\Delta s = a\Delta\theta$, is $\Delta T \cos \frac{1}{2}\Delta\theta$. Hence, denoting the *intensity* of friction at any point by $F$, we have, proceeding to the limit,

$$F = \left.\frac{\Delta T \cos \frac{1}{2}\Delta\theta}{a\Delta\theta}\right]_{\Delta\theta = 0} = \frac{dT}{ad\theta}. \quad . \quad . \quad . \quad . \quad (2)$$

Now, if the excess of $T_1$ over $T_2$ is such that the cord is on the point of slipping from $B$ toward $A$, the limiting amount of friction is acting at every point, that is,

$$F = \mu R.$$

Hence, from equations (1) and (2),

$$\frac{dT}{ad\theta} = \mu\frac{T}{a},$$

or

$$\frac{dT}{T} = \mu d\theta.$$

In this expression $\theta$ is measured from $B$ toward $A$; hence integrating between limits, we have

$$\log T_1 - \log T_2 = \mu\theta,$$

whence

$$T_1 = T_2 e^{\mu\theta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This formula shows that the ratio of the tensions depends only upon the coefficient of friction and the angular measure of the arc of contact, and is therefore *independent of the size of the cylinder.*

**215.** When a large tension $T_1$ is to be sustained and the force $T_2$ available is small, friction is taken advantage of by taking several turns about a rough cylindrical post. If $n$ is the number of turns, we have, putting $\theta = 2n\pi$,

$$T_1 = T_2 e^{2n\mu\pi},$$

and, taking common logarithms,

$$\log_{10} T_1 = \log_{10} T_2 + 2.7288n\mu, \quad . \quad . \quad . \quad . \quad (4)$$

in which the constant is the value of $2\pi \log_{10} e$. Thus, for example, if three turns of a rope under the tension $T_1$ be taken around the post, and a force $T_2 = 100$ pounds be applied to the other end of the rope, it will not *surge*, or slip upon the post, unless $T_1$ is greater than the value determined by equation (4). Supposing $\mu = \frac{1}{4}$, this equation becomes

$$\log_{10} T_1 = 2 + 2.0466,$$

whence we find

$$T_1 = 11{,}133 \text{ pounds.}$$

**216.** When the arc of contact of the cord and surface is not circular, equation (1) becomes

$$R = \frac{T}{\rho},$$

where $\rho$ is the radius of curvature. In equation (2), $\rho d\phi$, which is the value of $ds$, takes the place of $ad\theta$. Thus

$$F = \frac{dT}{\rho d\phi}$$

and the final result is

$$T_1 = T_2 e^{\mu\phi}.$$

Since the variable $\rho$ has disappeared before integration, it appears that the ratio of the tensions is independent of the shape of the rough surface, depending only upon the coefficient of friction and the angle $\phi$, which is the total change of direction which the rope undergoes.

## EXAMPLES. XI.

1. On a rough plane of inclination $\theta$ the greatest value of the force acting along the plane and producing equilibrium is double the least. What is the coefficient of friction? $\mu = \frac{1}{3}\tan\theta$.

2. Two unequal weights, $W_1$ and $W_2$, on a rough inclined plane are connected by a string which passes through a smooth pulley in the plane. Find the greatest inclination of the plane consistent with equilibrium.
$$\tan\theta = \frac{W_1 + W_2}{W_1 - W_2}\mu.$$

3. Two rough bodies, $W_1$ and $W_2$, rest upon an inclined plane and are connected by a string parallel to the plane. If the coefficient of friction is not the same for both, determine the greatest inclination consistent with equilibrium, and the tension of the string.
$$\tan\theta = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}; \quad T = \frac{W_1 W_2}{W_1 + W_2}(\mu_1 - \mu_2)\cos\theta.$$

4. If the angle of friction is 30°, what is the least force which will sustain a weight of 100 pounds on a plane whose inclination is 60°? 50 pounds.

5. A uniform pole leans against a smooth vertical wall at an angle of 45° with it, the lower end being on a rough horizontal plane and about to slide. What is the value of $\mu$? $\mu = \frac{1}{2}$.

6. Two equal uniform beams, connected at one end of each by a smooth hinge, rest in a vertical plane with their other ends on a rough horizontal plane. If $\beta$ is the greatest possible angle at the hinge, what is the coefficient of friction?
$$\mu = \tfrac{1}{2}\tan\tfrac{1}{2}\beta.$$

7. A heavy uniform rod, whose length is $2a$, is supported on a rough peg, a string of length $l$ being attached to one end of the rod and fastened to a point in the same horizontal plane with the peg. If, when the rod is on the point of slipping, the string is perpendicular to it, show that $l = \mu a$.

8. A weight $W$ on a rough horizontal plane is attached to a string which passes over a smooth pulley at the height $a$ above

the plane and carries a weight $P$ hanging freely. It is found that $l$ is the least length of string between $W$ and the pulley consistent with equilibrium. What is the coefficient of friction?

$$\mu = \frac{P\sqrt{(l^2 - a^2)}}{Wl - Pa}.$$

9. Two equal rings of weight $W$ rest on a rough horizontal rod; a string of length $l$ passes through them and has both ends attached to a weight $W'$. If $\mu$ is the coefficient of friction for the rod and rings and there is no friction between the string and rings, what is the greatest possible distance between the rings?

$$\frac{l}{2}\left(1 - \frac{W'^2}{\mu^2(2W + W')^2}\right).$$

10. A uniform plank of weight $W$, length $l$, and whose thickness may be neglected, rests horizontally on a rough cylinder whose radius is $a$. Find the weight $W'$ which can be suspended from one end without causing the plank to slide, $\alpha$ being the angle of friction.

$$W' = \frac{2a\alpha}{l - 2a\alpha}W.$$

11. A hemisphere is supported by friction against a vertical wall and a horizontal plane of equal roughness. Find $\theta$, the greatest possible inclination of the plane base to the horizon.

$$\sin\theta = \frac{8\mu(1 + \mu)}{3(1 + \mu^2)}.$$

12. Three equal hemispheres rest with their circular bases upon a rough horizontal plane and tangent to one another. They support a smooth sphere of the same material and radius. What is the least possible value of $\mu$? $\mu = \frac{1}{6}\sqrt{2}.$

13. Show that, on a rough inclined plane, the locus of the extremities of lines representing forces which can be applied to a heavy particle along the plane without producing or permitting motion is a circle; and that, when motion begins to take place, it will be in a direction parallel to the corresponding radius of this circle.

14. On a rough plane inclined at the angle $\theta$ it was found that the least angle which a force acting along the plane and

sustaining a weight could make with the horizontal line in the plane was 60°. What was the coefficient of friction?

$$\mu = \tfrac{1}{2}\tan\theta.$$

15. Find the greatest horizontal force along the inclined plane, when $\theta < \alpha$, which can be applied to a weight $W$ without producing motion.

$$\frac{W}{\cos\alpha}\sqrt{(\sin^2\alpha - \sin^2\theta)}.$$

16. A weight $W$, resting upon a rough plane inclined at an angle of 30°, is attached to a string which passes in a horizontal direction parallel to the plane over a pulley, and supports a weight $\frac{1}{2}W\sqrt{2}$ hanging freely. If $W$ is on the point of moving, determine the coefficient of friction, and $\phi$, the angle between the string and the direction of motion.

$$\mu = 1\,;\ \sin\phi = \tfrac{1}{3}\sqrt{3}.$$

17. A uniform rod rests wholly within a hemispherical bowl in a vertical plane through its centre, and there subtends the angle $2\beta$. $\alpha$ being the angle of friction, determine $\theta$, the inclination of the rod to the horizon in limiting equilibrium.

$$\tan\theta = \frac{\sin 2\alpha}{2\cos(\beta+\alpha)\cos(\beta-\alpha)}.$$

18. Two weights, $P$ and $Q$, of the same material rest on a double inclined plane and are connected by a string passing over a smooth pulley at the common vertex, $\theta$ and $\psi$ being the inclinations of the planes, and $\alpha$ the angle of friction; $Q$ is on the point of motion down the plane. Show that the weight which may be added to $P$ without producing motion is

$$P\frac{\sin 2\alpha\sin(\psi+\theta)}{\sin(\theta-\alpha)\sin(\psi-\alpha)}.$$

19. Denoting $AG$ in Fig. 70 by $a$, and $GB$ by $b$, what is the least coefficient of friction that will allow the rod to rest in all positions?

$$\mu = \sqrt{\frac{a}{b}}.$$

20. If, in example 19, $\mu$ is the coefficient of friction between the rod and the ground, and $\mu'$ that between the rod and the

wall, show that the rod will rest in all positions if $\mu\mu'$ is not less than $\frac{a}{b}$.

21. If one cord of a balanced window-sash, whose height is $a$ and breadth $b$, is broken, what is the least coefficient of friction in order that the other weight may support the window?

$$\mu = \frac{a}{b}.$$

22. A cubical block stands upon a rough inclined plane and is attached to a fixed point by a cord passing from the middle of the upper edge, which is horizontal, in a direction perpendicular to it and parallel to the plane. Determine the greatest inclination for which the block will stand. $\tan\theta = 1 + 2\mu$.

23. A uniform heavy plank $AB$ rests with the end $A$ on a rough horizontal plane, and a point $C$ of its length touching a rough heavy sphere whose point of contact with the plane is $D$. Prove that the magnitude of the friction is the same at each of the points $A$, $C$ and $D$. If the coefficient of friction is the same at each point, and is diminished until slipping takes place, show that it will occur at $A$ or at $C$, according as $A$ and $D$ lie on the same or on opposite sides of the vertical through $B$.

24. A uniform beam $AB$ of weight $W$ lies horizontally upon two transverse horizontal beams at $A$ and $C$; a horizontal force $P$ at right angles with $AB$ is then applied at $B$ and is gradually increased until motion takes place. Putting $AB = 2a$ and $AC = b$, show that, if $3b > 4a$, slipping will take place at $C$ when $P = \frac{1}{2}\mu W$; and if $3b < 4a$, slipping will take place at $A$ when $P = \mu W \frac{b-a}{2a-b}$.

25. A man, by taking $2\frac{1}{2}$ turns around a post with a rope and holding back with a force of 200 pounds, just keeps the rope from surging. Supposing $\mu = 0.168$, find the tension at the other end of the rope. 2800 pounds.

26. A hawser is subjected to a stress of 10,000 pounds. How many turns must be taken around the bitts, in order that a man who cannot pull more than 250 pounds may keep it from surging, supposing $\mu = 0.168$? $3\frac{1}{2}$.

27. A weight of 5 tons is to be raised from the hold of a steamer by means of a rope which takes $3\frac{1}{2}$ turns around the drum of a steam-windlass. If $\mu = 0.234$, what force must a man exert on the other end of the rope? 65 pounds.

28. A weight of 2000 pounds is to be lowered into the hold of a ship by means of a rope which passes over and around a spar lashed across the hatch-coamings so as to have an arc of contact of $1\frac{1}{4}$ circumferences. If $\mu = \frac{7}{22}$, what force must a man exert at the end of the rope to control the weight? 164 pounds.

29. A weight is supported on an inclined plane by a cord parallel to the plane. If the cord can just sustain the weight when the plane is smooth and the inclination $45°$, what is the greatest possible inclination if $\mu = 1$? $75°$.

30. A uniform ladder weighing 100 pounds, and 52 feet long, rests against a rough vertical wall and a rough horizontal plane, making an angle of $45°$ with each. If the coefficient of friction is at each end $\frac{2}{3}$, how far up the ladder can a man weighing 200 pounds ascend before the ladder begins to slip? 47 feet.

31. A heavy homogeneous hemisphere rests with its convex surface on a rough inclined plane. If the inclination be gradually increased, the hemisphere will roll until it either slides or tumbles over. If $\mu = \frac{2}{5}$, will it tumble or slide?

# CHAPTER VI.

## FORCES IN GENERAL.

### XII.

### Lines of Action neither Coplanar nor Parallel.

**217.** When the lines of action of two forces acting on a rigid body lie in a single plane, they either intersect or are parallel; and we have seen that, in either case, we can find a single force whose action is equivalent to the joint action of the two given forces. Hence, in the case of a coplanar system of forces, and also in the case of a system of parallel forces, we were able, by combining the forces two by two, to reduce the joint action of the system to that of a single force, called *the resultant*, except when the final pair of forces happen to form *a couple*.

But, when the lines of action of two given forces do not lie in one plane (that is to say, neither intersect nor are parallel), *there is no single force whose action is equivalent to the joint action of the two forces.*

We proceed, in this section, to analyze the joint action of a system of forces in general, and shall find that the simplest mechanical equivalent of such a system consists of a force together with a couple.

### The Moment of a Force about any Axis.

**218.** In the preceding chapters, the moment of a force about an axis has been defined only in the case of an axis perpendicular to (though not intersecting) the line of action. In other

words, supposing the solid upon which the forces act to be free to turn about a fixed axis, we have considered the turning moment, or tendency to turn about the axis, produced by a force whose line of action lies in some plane perpendicular to the axis. We have now to consider the turning effect of a force whose line of action is oblique to the axis.

Let $CO$, Fig. 72, be the axis, and $P$ the force acting at $A$ in the line $AB$, which does *not* lie in the plane $MN$ passing through $A$ and perpendicular to the axis. Draw $AE$ parallel to the axis, and let $AD$ be the intersection of the plane $EAB$ with the plane $MN$ which cuts the axis in $O$. $AD$ is then the projection of $AB$ upon this plane, the projecting plane $BAD$ being perpendicular to $MN$. Now let $P$ be resolved into rectangular components acting in the lines $AE$ and $AD$. The first of these components, being parallel to the axis, obviously has no tendency to turn the body about the axis. Hence the turning effect of the force $P$ is entirely due to the component $AF$ along $AD$. We therefore *define* the moment of $P$ about the axis $CO$ to be the same thing as the moment of the component $AF$ in a plane perpendicular to the axis.

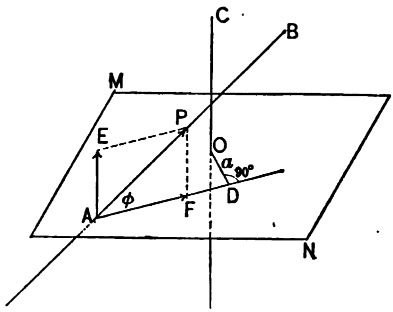

FIG. 72.

Denoting this moment by $H$, the inclination of the line of action to the plane just mentioned (or angle $BAD$ in the figure) by $\phi$, and the distance from $O$ to the projected line of action by $a$, we have $AF = P \cos \phi$, and therefore

$$H = aP \cos \phi.$$

**219.** It will be noticed that $\phi$ and $a$ will have the same values wherever the point of application $A$ be taken on the line

of action of $P$. The distance $a$ may be defined as the distance between the axis and a plane parallel to it through the line of action, or as *the distance between two parallel planes passing each through one of these lines and parallel to the other*, or finally as *the common perpendicular to* (or shortest distance between) *the axis and the line of action.*

If $\theta$ denotes the angle $EAB$, in Fig. 72, which is the complement of $\phi$, we have

$$H = aP \sin \theta,$$

where $\theta$ is the inclination of the line of action to the axis (or angle between any intersecting lines parallel to them), and $a$ the shortest distance between them.

**220.** The combined turning effect of two forces about a given axis, is the sum or difference of the moments of the forces according as they tend to turn the body in the same or in opposite directions. In like manner, adopting one direction of rotation as positive and the opposite as negative, it is readily seen that the joint turning effect of a system of forces, or *resultant moment* of the system about a given axis, is *the algebraic sum of the moments of the several forces.*

In the case of a system of forces in equilibrium, the resultant moment must vanish for every axis.

### Representation of a Couple by a Vector.

**221.** We have seen in Art. 97 that the moment of a couple is the same as the algebraic sum of the moments of the two forces which constitute the couple about any point whatever in the plane of the couple, that is to say, about *any* axis perpendicular to the plane of the couple. Such an axis is called *the axis of the couple*; and, since we have seen in Art. 144 that couples of the same moment in parallel planes are equivalent, it appears that the direction of the axis and the magnitude of the moment are the only essential features of a couple. Provided these are

given, the *position* of the plane and that of the axis are immaterial.

It follows that, denoting the moment of the couple by $H$, *a length representing H on any given scale laid off upon an axis of the couple will, by its magnitude and direction, completely represent the couple.* In doing this, one direction upon the axis will be chosen to represent a particular direction of rotation about the axis. For example, the couple in the plane $MN$ shown in Fig. 73 is usually represented by a length $AE$ measured from $A$ in the plane of the couple toward $O$, where $O$ is on that side of the plane from which the direction of rotation produced by the couple appears as positive or counter-clockwise.

## Moment of a Couple about an Oblique Axis.

**222.** Consider now the turning effect of the couple $H$ in the plane $MN$ about an axis oblique to its plane. Let $AC$, Fig. 73, be the oblique axis cutting the plane $MN$ at $A$, and take $AO$, the perpendicular to the plane at this point, as the axis of the couple. Denote by $\phi$ the angle $OAC$ between the axes, and put $H = aP$. Draw $AD$ the projection of $AC$ upon the plane $MN$, and $AB = a$ perpendicular to it in the plane $MN$. Then the force $P$ acting at $B$, parallel to $AD$ and in the proper direction, will have a moment about $AO$ equal to that of the given couple $H$; and this force, together with $P$ acting in the opposite direction at $A$, will represent the couple.

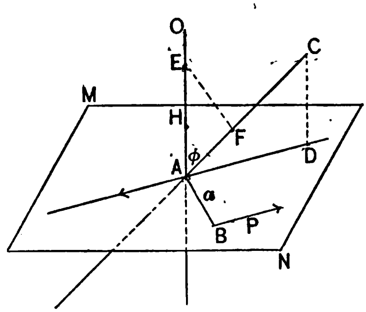


FIG. 73.

Now, since $AB$ is the common perpendicular to $AC$ and the line of action of $P$, and $\phi$ is the complement of their inclination, the moment of $P$ about $AC$ is

$$aP \cos \phi$$

(in fact $P\cos\phi$ is the resolved part of $P$ in a direction perpendicular to $AC$). Now $P$ acting at $A$ has no moment about $AC$, hence $aP\cos\phi$ is the entire moment of the couple about $AC$. That is, since $H=aP$, the moment of the couple $H$ about an axis inclined to its axis at the angle $\phi$ is $H\cos\phi$.

### Resolved Part of a Couple.

**223.** The axis of the couple is sometimes called its *principal axis,* in contradistinction to an oblique axis about which its moment may be considered. When a line $AE$ representing $H$, as explained in Art. 221, is measured off on the principal axis (see Fig. 73), the projection $AF$ of this line on the oblique axis is $H\cos\phi$, which we have seen is the moment about the oblique axis. Thus *the effective part* of the couple $H$ in producing rotation about the oblique axis is equivalent to a couple whose moment is $H\cos\phi$, and *is represented by the projection of the line representing $H$ upon the new axis,* exactly as the effective part of a force $P$ in a given direction is represented by the resolved part of the line representing $P$.

It should be noticed that $\phi$ is the angle between the directions taken as positive along the two axes; if the opposite direction were regarded as positive on the oblique axis, the angle $\phi$ would be obtuse, and $H\cos\phi$, the resolved part of the couple, would be negative.

### Composition of Couples.

**224.** It follows from the preceding articles that, if we have any number of couples about axes in different directions, their joint moment about any axis is represented by the sum of the projections of the lines representing the couples. These lines, having direction and magnitude only, are simple vectors, and if they be added vectorially, as is done for forces in Fig. 15, p. 43, the sum of the projections will be the projection of the vectorial sum. Hence the joint effect of a system of couples in producing rotation about a given axis is the same as that of the couple represented by the vectorial sum of the vectors representing the

given couples. Since this is true for every direction of the given axis, the couple just mentioned is the exact equivalent of the system of couples and is therefore called their *resultant.*

In the case of two couples, this agrees with what is proved in Art. 146, for the axes of the planes of the given couples and their resultant evidently make plane angles equal to the diedral angles made by the corresponding planes.

### Joint Action of a System of Forces.

**225.** We have seen in Art. 102 that, given a force $P$ and a selected point $A$ not in the line of action, we may, by assuming two equal and opposite forces acting in a parallel line at $A$ (see Fig. 36), replace the force $P$ by an equal parallel force at $A$ together with a couple. The plane of this couple is that containing $A$ and the line of action of $P$, and its moment is the moment of $P$ about $A$.

Suppose now that this is done for each of the forces of a given system. We shall then have replaced the whole system of forces by a system of equal and parallel forces acting at $A$, together with a system of couples in different planes. The forces at $A$ may be combined, by vectorial addition, into a resultant force acting at $A$, and the couples may, in accordance with the preceding article, be combined in like manner into a single resultant couple $K$.

Thus the whole system of forces has been replaced by *a force $R$ acting at a selected point $A$ together with a couple $K$.* The plane of the couple $K$ will *not* in general be parallel to the line of action of $R$, so that the force and couple cannot be replaced, as in Art. 101, by a single force.

**226.** This combination of a force and a couple which cannot be reduced to a single force is called a *dyname.* We have thus found that the resultant of a system of forces is, in general, not a single force, but a dyname.

Since the magnitude and direction of $R$ are *vectorially* determined from the given forces, they are independent of the position of the selected point $A$, so that the *resultant force-vector* is

constant. But the moment of the couple $K$ and the direction of its axis depend upon the position of $A$. For, if $A$ be moved to a point $B$ not on the line of action of $R$, the effect is to combine with $R$ acting at $A$ a couple (as in Art. 101); then the reverse couple, which must be combined with $K$, will make an alteration in the value of $K$. The dyname consisting of $R$ acting at $A$ and the couple $K$ may be denoted by $(R_A, K)$.

### The Principal Moment of a System at a Point.

**227.** Since the given system of forces is equivalent to the dyname $(R_A, K)$, the moment of the system about any axis passing through $A$ is the moment of $K$ about that axis, because $R$ acting at $A$ has no moment about any axis passing through $A$. It follows that $K$ is the greatest value of the moment of the system about any axis passing through $A$. It is therefore called the *principal moment* of the system at $A$, and its axis, or principal axis (see Art. 223), is called *the principal axis of moment at* $A$. It follows that the moment of the system about an axis making the angle $\phi$ with the principal axis is $K \cos \phi$, exactly as in the case of a single force or of a couple.

### Poinsot's Central Axis.

**228.** In Fig. 74, let $AB$ represent the resultant force $R$ of the system acting at $A$, and let $AC$ represent, as in Art. 221, the axis and magnitude of the resultant couple $K$. Draw $AE$ perpendicular to $AB$ in the plane $BAC$, and let the couple $K$ be resolved into rectangular components whose axes are $AD$, in the direction of the line of action of $R$, and $AE$ perpendicular to it. Denote by $G$ the first of these couples, and by $\psi$ the angle $BAC$. Then, by Art. 223,

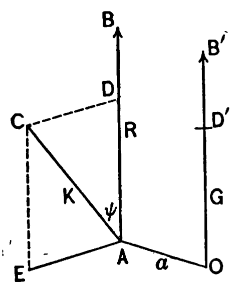


FIG. 74.

$$G = K \cos \psi. \quad . \quad . \quad . \quad (1)$$

Draw $AO$ perpendicular to the plane $BAC$; then the plane of

the other component couple whose axis is $AE$ is the plane $BAO$, containing the line $AB$. This couple, whose value is $K \sin \psi$, can therefore be combined with the force $R$ by the method of Art. 101. For this purpose determine $a$ so that

$$aR = K \sin \psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and lay off $AO = a$ on that side of $A$ which makes the moment of $R$ acting at $O$ about the axis $AE$ agree in direction with the couple $K \sin \psi$. Then $R$ acting at $O$ is equivalent to $R$ acting at $A$ together with the couple $AE$. Hence the whole system of forces is equivalent to the dyname consisting of the force $R$ acting at $O$ and the couple $G$ *whose axis is in the direction of the line of action of* $R$.

**229.** The line of action of $R$ when the system is thus reduced to the dyname $(R, G)$, in which the line of action is also the axis of the couple, is known as *Poinsot's Central Axis;* and the dyname of this character has been called a *wrench.*

The central axis of a system of forces has a definite position independent of the position of the initial point $A$. For, suppose it possible that the system could be reduced to another wrench $(R', G')$, where $R'$ has a different line of action from $R$. Since its direction is the same as that of $R$, it has a parallel line of action, and the axis of $G'$ is the same as that of $G$, that is, $G$ and $G'$ are couples in the same plane. Now by combining $R'$ and $G'$ each reversed with the system $(R, G)$, we shall have a system in equilibrium. Therefore the couple formed by $R$ and $R'$ reversed is in equilibrium with the couple $G - G'$; but this is impossible unless they both vanish, because these couples are in different planes. Hence $R$ and $R'$ act in the same line, and $G' = G$.

**230.** Equation (1), Art. 228, shows that $G$ is less than any other value of $K$; so that the central axis is the locus of the points for which the principal moment is a minimum.

Supposing $R$, $G$ and the central axis to be known, to determine the principal moment $K$ and the direction of the principal

axis at any point $A$, let $a$ be the distance of $A$ from the central axis; then from equations (1)and (2) we derive

$$K = \sqrt{(a^2R^2 + G^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\tan\psi = \frac{aR}{G}.^* \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

**231.** If, for a given system of forces, we find the force-vector $R$ reduces to zero, the dyname reduces to the couple $K$, which in this case will be independent of the position of $A$.

If, on the other hand, we find $K = 0$, the dyname reduces to a single force $R$ acting at $A$.

But the general condition that the dyname, or resultant of the system of forces, should reduce to a single force is that $G$ shall vanish. This occurs, according to equation (1), Art. 228, not only when $K = 0$, but when

$$\psi = 90°;$$

in other words, when for any selected point $A$ the axis of the couple $K$ is perpendicular to the line of action of $R$. The pro-

* Referring to Fig. 74, it follows that, for any point $A$ on the cylindrical surface whose axis is the central axis and whose radius is $a$, the axis of the couple $K$ or principal axis is tangent to a spiral described on the surface, making with the elements the constant angle $\psi$ determined by equation (4).

The portion of an element intercepted between two whorls of this spiral is

$$2\pi a \cot\psi = \frac{2\pi G}{R},$$

which is independent of $a$; therefore every such spiral is the intersection of a cylinder with a helical or screw surface whose pitch is $\frac{2\pi G}{R}$.

cess in Art. 228 then gives the line of action of the single force which is the resultant of the system.

## Forces Referred to Three Rectangular Axes.

**232.** In referring a system of forces to three rectangular axes, we shall take them in such a manner that positive rotation about the axis of $z$ (that is, positive rotation in the plane of $xy$ as viewed from the side on which $z$ is positive) shall be rotation from the positive direction of the axis of $x$ to that of the axis of $y$, as in Fig. 75. It follows that positive rotation about the axis of $x$ is rotation from $y$ to $z$, and that about the axis of $y$ is rotation from $z$ to $x$.*

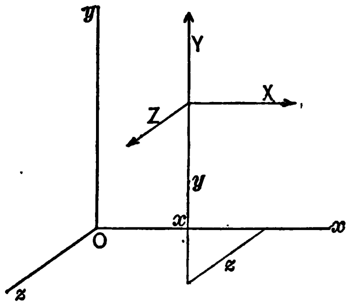


FIG. 75.

Let $(x, y, z)$ be the point of application, and $X$, $Y$, $Z$ the resolved parts, in the direction of the axes, of a force $P$. The moment of $P$ about the axis of $x$ is the algebraic sum of the moments of the resolved parts $Y$ and $Z$, since $X$ which is parallel to the axis of $x$, has no moment about it. The moment of $Z$ about the axis of $x$, to which it is perpendicular, is $yZ$, since $y$ is the common perpendicular to the axis and the line of action. This moment is positive, because, when $y$ and $Z$ are positive as in the figure, $Z$ tends to turn the ordinate $y$ toward the positive direction of the axis of $z$. In like manner, the moment of $Y$ about the axis of $x$ is $zY$, but this moment is found to be negative. Hence, denoting the moment of $P$ about the axis of $x$ by $L$, we have

$$L = yZ - zY. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* The diagrams being drawn as if the observer were situated in the first octant, the letters $x, y, z$ appear to follow one another in positive rotation about the origin.

Similarly, denoting the moments about the axis of $y$ and $z$ by $M$ and $N$ respectively, we have

$$M = zX - xZ, \quad \dots \dots \quad (2)$$
$$N = xY - yX. \quad \dots \dots \quad (3)$$

**233.** The six quantities $X$, $Y$, $Z$, $L$, $M$ and $N$ may be taken as the determining elements or coordinates of a given force, and each of these quantities has a definite value for a given force; but it can be shown that they are not six *independent* elements.

For, suppose these six quantities to be given in equations (1), (2) and (3); if $x, y$ and $z$ (regarded now as unknown quantities) admit of any actual values, we shall have, by multiplying the equations by $x, y$ and $z$ respectively and adding,

$$LX + MY + NZ = 0. \quad \dots \dots \quad (4)$$

This is therefore a necessary relation which must exist between the six elements of a force. If it does not hold true, it is impossible to find values of $x, y$ and $z$, the equations being, in that case, inconsistent. But, if it does hold true, the equations will not determine *definite values* of $x, y$ and $z$; they are, in that case, the equations of three planes which intersect in one line, and this line is the line of action of the force. Thus, as we should expect, the point of application is not determined, but only the line of action. Any two of the equations (1), (2) and (3) may be taken as *the equations of the line of action.*

## Six Independent Elements of a System of Forces.

**234.** The advantage of employing the six elements $X$, $Y$, $Z$, $L$, $M$, $N$ arises from the fact that the joint effect of a system of forces is found by simply adding the like elements of the several forces. Thus, in the case of a system of forces $P_1, \dots P_n$, put

$$\left.\begin{array}{lll} X' = \Sigma X, & Y' = \Sigma Y, & Z' = \Sigma Z, \\ L' = \Sigma L, & M' = \Sigma M, & N' = \Sigma N; \end{array}\right\} \quad \dots \quad (1)$$

then $X'$, $Y'$, $Z'$, $L'$, $M'$, $N'$ constitute the like elements of the total resultant of the system. Moreover, these are *six independent elements* which may have any values whatever, *not* generally satisfying a relation like equation (4) of the preceding article. It is only when they happen to satisfy such an equation that there exists a single force equivalent to the system.

**235.** In the general case, suppose the origin to be taken as the selected point of Art. 225; then we have seen that the system is equivalent to a force $R$ acting at the origin together with a couple $K$. Since $R$ at the origin has no moment about either axis, $L$, $M$ and $N$ are the moments of $K$ about the three axes respectively. Therefore, by Art. 223, they are the resolved parts of the couple $K$ about the axes, and the axial representations of them are the resolved parts or projections of the vector $K$, just as $X$, $Y$ and $Z$ are the projections of the vector $R$. Let $\alpha$, $\beta$, $\gamma$ be the direction angles of $R$, so that

$$R = \sqrt{(X'^2 + Y'^2 + Z'^2)}, \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and

$$\cos\alpha = \frac{X'}{R}, \qquad \cos\beta = \frac{Y'}{R}, \qquad \cos\gamma = \frac{Z'}{R}. \quad . \qquad (3)$$

These equations determine the magnitude and direction of $R$. Similarly if $\lambda$, $\mu$, $\nu$ are the direction angles of $K$, we have

$$K = \sqrt{(L'^2 + M'^2 + N'^2)}, \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

$$\cos\lambda = \frac{L'}{K}, \qquad \cos\mu = \frac{M'}{K}, \qquad \cos\nu = \frac{N'}{K}, \quad . \qquad (5)$$

which determine the magnitude and direction of $K$.

**236.** To find the value of $G$, the minimum couple, which is associated with $R$ acting in the central axis, let $\psi$ denote the angle between the directions of $R$ and $K$, as in Fig. 74; then $G$, the projection of $K$ upon $R$, is the sum of the projections of $L'$, $M'$ and $N'$ (compare Art. 61). Hence,

$$G = K\cos\psi = L'\cos\alpha + M'\cos\beta + N'\cos\gamma;$$

or, substituting the values in equations (3) and (2),

$$G = \frac{L'X' + M'Y' + N'Z'}{\sqrt{(X'^2 + Y'^2 + Z'^2)}}. \quad . \quad . \quad . \quad . \quad (6)$$

The condition that the resultant of the system may be a single force is that $G$ shall vanish; hence it is

$$L'X' + M'Y' + N'Z' = 0,$$

which agrees with the result found in Arts. 233 and 234. Conversely, if this condition is satisfied, the resultant must be either a single force or a couple.

237. To determine the central axis, we observe that, because $R$ in the central axis and the couple $G$ are together equivalent to the given system of forces, the sum of their moments about the axis of $x$ must be $L'$. The moment of $G$ about that axis is $G \cos \alpha$; therefore that of $R$ in the central axis is

$$L' - G \cos \alpha,$$

and, in like manner, the moments about the axes of $y$ and $z$ are

$$M' - G \cos \beta \qquad \text{and} \qquad N' - G \cos \gamma.$$

Hence, by Art. 233, any two of the equations

$$\left. \begin{aligned} yZ' - zY' &= L' - G \cos \alpha, \\ zX' - xZ' &= M' - G \cos \beta, \\ xY' - yX' &= N' - G \cos \gamma, \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (7)$$

determine the line of action of this force; that is to say, these are the equations of the central axis.

## Conditions of Equilibrium.

**238.** The system of forces is in equilibrium when the six elements of the resultant $X'$, $Y'$, $Z'$, $L'$, $M'$, $N'$ all vanish; that is, when

$$\Sigma X = 0, \qquad \Sigma Y = 0, \qquad \Sigma Z = 0,$$

$$\Sigma L = 0, \qquad \Sigma M = 0, \qquad \Sigma N = 0.$$

Thus, when the forces of a system in equilibrium are unrestricted, there are six independent conditions of equilibrium which must be fulfilled; and from these it is possible to determine six, and not more than six, unknown quantities. The equations above are the simplest form of the conditions of equilibrium when the forces are referred to coordinate axes; but a condition of equilibrium can, of course, be found by resolving forces in *any* direction, or by taking moments about *any* axis.

The conditions obtained by resolving forces are precisely the same as if the forces all acted at a single point; hence, as in Art. 74, only three independent conditions can be found in this way, and in order to be independent the three directions of resolving must not lie in one plane. It follows that three at least of the six independent conditions must be derived by taking moments.*

**239.** It is possible, however, to obtain all the conditions of equilibrium by taking moments about different axes; but, in order that they should be independent, there are some restrictions upon the choice of these axes. For example, if two of the axes intersect in a point $A$, the vanishing of the moment about a third axis passing through $A$ and in the plane of the two axes will *not* give an independent condition. If the third axis passes through $A$ but is not coplanar with the other two, it gives an independent condition. Again, in this last case, the moment

*In like manner we have seen in Art. 109 that at least one of the three conditions, when the forces are restricted to a given plane, must be derived from the principle of moments.

necessarily vanishes about any other axis passing through $A$, so that a fourth axis passing through $A$ would not give an independent condition.

## Equilibrium of Constrained Bodies.

**240.** Suppose a rigid body to have its possible motions limited or constrained by means of fixed bodies with which it is in contact. This may be done, for example, by having one or more of its points fixed or confined to fixed surfaces or lines, or by having its surface in contact with a fixed surface. If such a body be acted upon by external forces, it will in general move subject to the constraints; and, if it is at rest, the external forces together with the resistances of the fixed bodies must form a system of forces in equilibrium.

Let $n$ denote the smallest number of numerical elements which will serve to determine the unknown resistances or forces producing the constraint. Since the whole number of unknown quantities is six, $n$ must be less than six; therefore, if these $n$ unknown quantities were eliminated from the six equations of equilibrium, there would remain $6 - n$ equations *independent of the forces of constraint*, which are therefore conditions imposed upon the external forces in order that equilibrium may exist. These equations, whether found by elimination or directly by a method which will be given in the following section, are called *the conditions of equilibrium for the constrained body.*

**241.** If we put $6 - n = m$, the body thus constrained in its motion is said to be subject to $n$ degrees of constraint, and to possess $m$ *degrees of freedom*. Thus, the perfectly free rigid body has 6 degrees of freedom. The constrained body with $m$ degrees of freedom requires the knowledge of the values of $m$ numerical determining quantities or elements to fix its position; and, if the external forces are either given quantities or known functions of these $m$ elements, the latter will be the unknown quantities to be determined by means of the $m$ conditions of equilibrium.

In the case of a free *particle*, there are but three degrees of freedom and accordingly three determining elements or coordi-

nates fix its position. But, in the case of a rigid body, after a point $A$ of the body is fixed, the body still has three degrees of freedom. This may be clearly seen as follows: When $A$ is fixed, a point $B$ of the body at a given distance from $A$ is thereby restricted to the surface of a given sphere. Two determining elements or coordinates are therefore necessary to determine the position of $B$. But, after $B$ is fixed as well as $A$, the body is still free to turn about the axis $AB$. A point $C$, not in the line $AB$, is now restricted to a given circle; and therefore one more determining element will fix it, and thus completely determine the position of the rigid body.

**242.** It will be noticed that, in the illustration above, the fixing of the point $A$ is equivalent to three degrees of constraint. The body retains three degrees of freedom, and three equations of equilibrium are necessary. The remaining three of the six conditions of equilibrium of the general case would serve to determine the resistance at $A$, which, being unknown in magnitude and direction, involves *three* unknown elements.

Again, when two points are fixed, so that the body rotates about a fixed axis, the body retains but one degree of freedom, and but one condition of equilibrium is necessary. The remaining five conditions of the general case would serve to determine the reactions of the axis.

**243.** As a further illustration, if three points (not in a straight line) of a rigid body are constrained to remain in a given plane, we shall have $n = 3$, because (see Art. 240) three unknown quantities, namely, the values of the normal resistances or reactions of the plane at these three points, are sufficient to determine a set of forces capable of producing the constraint. Therefore the body will be subject to three degrees of constraint. Hence we have also $m = 3$: the body has three degrees of freedom, and three conditions of equilibrium are required.

The case is that of a body capable of *plane motion* only, just as if it were a lamina subject only to forces acting in its plane. If this plane is taken as that of $xy$, the conditions of equilibrium, in the standard form of Art. 238, reduce to three, namely,

$\Sigma X = 0$, $\Sigma Y = 0$ and $\Sigma N = 0$, which are independent of forces in the direction of the axis of $z$. These are identical with the conditions given in Art. 100 for coplanar forces.

## EXAMPLES. XII.

1. When a force is represented by a line $AB$, show that its moment about any axis through $O$ is represented by double the projection of the area $OAB$ on a plane perpendicular to the axis; also that, when a couple is represented by an area, the resolved part of the couple in any plane is represented by the projection of the area.

2. Show that four forces acting in the sides of a quadrilateral which is not plane, and represented by them taken in one continuous direction about the perimeter, are equivalent to a couple in a plane parallel to the two diagonals and represented by double the area enclosed by the projections of the sides on this plane.

3. Show directly that the forces constituting a couple in a plane and those constituting the reverse couple in a parallel plane are in equilibrium.

4. Show that lines laid off from $O$ representing the moment of a force, or of a system of forces, about different axes passing through $O$ form chords of a sphere which passes through $O$ and of which the diameter represents the principal moment.

5. If $P$ be the value of each of two equal forces, $2a$ the shortest distance between the lines of action, and $2\alpha$ the angle between their inclinations, show that the central axis bisects the distance and the angle, and determine $R$ and $G$.

$$R = 2P \cos \alpha; \quad G = 2aP \sin \alpha.$$

6. Prove that, if the moment of a system of forces about each side of a triangle vanishes, the resultant is either a force or a couple in the plane of the triangle.

7. Six equal forces act in consecutive directions along those edges of a cube which do not meet a given diagonal. Find their resultant. The couple $2Pa\sqrt{3}$.

8. A force 3 acts parallel to the axis of $z$ at the point (4, 3), and a force 4 acts in the negative direction in the axis of $x$. Determine the central axis and the values of $R$ and $G$.

$$\left.\begin{aligned} 25y &= 27, \\ 4z + 3x &= 12; \end{aligned}\right\} \quad R = 5; \; G = -\frac{36}{5}.$$

9. $OABC$ is a tetrahedron, of which the edges meeting at $O$ are mutually at right angles. Forces are represented in magnitude, direction and line of action by $OA$, $OB$, $OC$, $AB$, $BC$, $CA$. Taking the first three edges as axes, and equal $a$, $b$, $c$, show that the resultant is a force represented by the line joining the origin with the point $(a, b, c)$, and the couple $\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$ in the plane $ABC$. Determine also the value of $G$.

$$G = \frac{3abc}{\sqrt{(a^2 + b^2 + c^2)}}.$$

10. If $P$ and $Q$ are two forces whose directions are at right angles, show that the central axis divides the distance $a$ between their lines of action inversely in the ratio $P^2 : Q^2$ and that $G = \dfrac{aPQ}{\sqrt{(P^2 + Q^2)}}$.

11. An upper half port, whose weight, 48 pounds, acts at its middle point, is 36 inches long and 15 inches broad, and is held in a horizontal position by a laniard, the single part of which passes through a hole in the bulwark 8 inches above the hinges. Find the tension on the bridle, which is 39 inches long and secured to the corners of the port, and the total action on each hinge. 66.3 lbs.; 25.5 lbs.

12. A pair of "sheer legs" is formed of two equal spars lashed together at the tops, so as to form an inverted V. They stand with their "heels" 20 feet apart on the ground, and would be 40 feet high if vertical. They are supported in a position 12 feet out of the vertical by a guy made fast to a point in the ground 60 feet to the rear. Find the tension on the guy and the thrust on each leg when lifting a 30-ton gun.

$T = 12.8$ tons, $R = 19.5$ tons.

13. The legs of a pair of sheers are at an angle of 60° with each other, and the plane of the sheers is inclined 60° to the

horizontal; the supporting guy is inclined $30^\circ$ to the horizontal. Find the thrust on each leg when a weight of 20 tons is lifted.

20 tons.

14. The line of hinges of a door is inclined at an angle $\alpha$ to the vertical. Show that the couple necessary to keep it in a position inclined at an angle $\beta$ to that of equilibrium is proportional to $\sin\alpha \sin\beta$.

15. A load of 10 tons is suspended from a tripod whose legs are inclined $60^\circ$ to the horizontal. A horizontal force of 7 tons is applied at the top in such a manner as to produce the greatest possible thrust in one leg. Find in tons that thrust and the stress on each of the other legs. 13.18; − 0.82.

16. A square is formed of uniform rods of length $a$ and weight $W$, freely joined together. One rod being fixed in a horizontal position, find the couple required to turn the opposite rod through the horizontal angle $\theta$. $aW \sin \tfrac{1}{2}\theta$.

17. Each of two strings of the same length has one end fastened to each of two points, whose distance is horizontal and equal to $a$. A smooth sphere of radius $r$ and weight $W$ is supported upon them, the plane of each string making the angle $\alpha$ with the vertical. Find the tension of either string. $\dfrac{Wa}{8r\cos\alpha}$.

18. A rod of length $a$ can turn about one end in a horizontal plane. A string tied to the other end passes over a smooth peg at a distance $b$ vertically above it, and is then attached to a given weight. The rod is then turned through an angle $\theta$, and is kept in position by a horizontal force $P$, applied at the end of the rod perpendicularly to it. Prove that $P$ is a maximum when

$$\tan^4\frac{\theta}{2} = \frac{b^2}{b^2 + 4a^2}.$$

# CHAPTER VII.

## THE PRINCIPLE OF WORK.

### XIII.

### Work done by or against a Force.

**244.** When the point of application of a constant force $P$ is displaced in the direction of the force through a space $s$, the force is said to do *work*, and the product $Ps$ is taken as the measure of the work done. When the displacement is in the direction opposite to that of the force, work is said to be done *against the force*. The unit of work is a compound unit involving the unit of space or length and the unit of force; thus the ordinary unit of work is *the foot-pound*, which may be defined as the work done by the gravity of a pound descending through one foot, or the work done against gravity in lifting one pound through the space of one foot.

That part of Mechanical Science which deals with forces as overcoming resistances through definite spaces is known as *Dynamics*, in distinction from Statics, in which the points of application of the forces are regarded as fixed. Compare Art. 49.

**245.** If the displacement of the point of application takes place in a line oblique to the line of action of the force, while the force remains constant in direction as well as in magnitude, the work done by the force is defined as *the product of the force and the projection of the displacement upon the line of action.* Thus, in Fig. 76, let $AP = P$ represent the force, and let $AB = s$ be the displacement, making the angle $\phi$ with the direction of $P$.

Denote the projection $AC$ of $s$ upon the line of action by $p$. Then $Pp$ is the work done by the force during the displacement; and, since $p = s \cos \phi$, we may also take

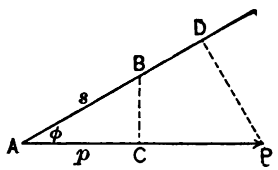

FIG. 76.

$$Ps \cos \phi$$

as the expression for the work done. It will be noticed that, when the angle $\phi$ is obtuse, $p$ has a direction opposite to that of the force, so that work is done against the force. In this case, the expression for the work becomes negative. Thus, work done *against* a force is regarded as *negative*.

**246.** The work done by a force in a given displacement oblique to the line of action is the same thing as the work done by the resolved part of the force in the direction of the displacement; for, in Fig. 76, this resolved part is $AD = P \cos \phi$; and, multiplying this by the displacement $s$, we have the work of the resolved part equal to $Ps \cos \phi$. Thus the "resolved part" of the force (see Art. 56) is the only "effective" part of the force in respect to work done. It will be noticed that the expression for the work done by the other component of the force in this case vanishes, because the angle between that component and the displacement is a right angle.

## Work done by the Components of a Force.

**247.** Let $R$ be the resultant of any number $P_1, P_2 \ldots P_n$ of forces acting at a single point of application which undergoes the displacement $s$ in any direction. Then, denoting the inclinations of $R, P_1, P_2 \ldots P_n$ to the direction of $s$ by $\phi, \phi_1, \phi_2, \ldots \phi_n$, we have, by resolving the forces in the direction of displacement,

$$R \cos \phi = P_1 \cos \phi_1 + P_2 \cos \phi_2 + \ldots + P_n \cos \phi_n. \qquad (1)$$

Multiplying by $s$, we have

$$Rs \cos \phi = P_1 s \cos \phi_1 + P_2 s \cos \phi_2 + \ldots + P_n s \cos \phi_n, \quad (2)$$

of which the several terms are, by Art. 245, expressions for work. Hence *the work of the resultant is equivalent to the algebraic sum of the works of the components.* The equation may be written in the form

$$Rp = P_1 p_1 + P_2 p_2 + \ldots + P_n p_n,$$

where $p, p_1, p_2 \ldots p$ are, as in Art. 245, the projected displacements taken in the directions of $R$, $P_1$, $P_2 \ldots P_n$.

**248.** Accordingly, when several forces act upon a particle, the total work of the forces is the algebraic sum of the works of the several forces, each reckoned independently of the existence of the others. Thus, if a weight $W$, Fig. 77, be displaced through the space $s$ up a plane inclined at the angle $\theta$, the total work is the sum of those of the several forces; namely, the weight $W$ acting vertically, the normal resistance $R$ and the frictional resistance $F$ (if the plane is rough). If $h$ is the vertical height through which the body is raised (so that $h = s \sin \theta$), $-Wh$ is the work of $W$, which is negative because $h$ is measured upward or *against* the force. No work is done either by or against the normal resistance $R$. The work of the frictional resistance $F$ is $-Fs$, because work is done against friction. Thus the total work is $-(Wh + Fs)$; that is, the work $Wh + Fs$ must be done against the forces in producing this displacement.

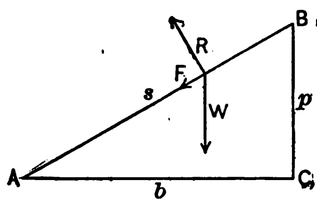

FIG. 77.

If the body were displaced down the plane, the work would be $Wh - Fs$; work would now be done by gravity, but against friction, because the latter force would now act up the plane.

## Virtual Work of a Variable Force.

**249.** When the force $P$ is variable either in magnitude or direction, if $ds$ is an element of displacement in the direction of the force, $Pds$ is the corresponding element of work. Again, if the element of displacement makes the angle $\phi$ with the direction of $P$, the element of work is $Pds \cos \phi$. In either case, the total work is an integral of this element.

A small displacement denoted by $\delta s$ and treated as an element is sometimes called a *virtual displacement*, and the corresponding expression,

$$P\delta s \cos \phi,$$

is called *the virtual work* of $P$ in this displacement. Thus, in the expression for virtual work, the magnitude of the force and the direction, both of the force and of the displacement, are regarded as constant.

It follows (see Art. 247) that, for any forces acting at a single point of application, the virtual work of the resultant is equal to the algebraic sum of the virtual works of the components. In particular, *if the forces are in equilibrium*, the resultant vanishes, and therefore *the algebraic sum of the works of the forces in any virtual displacement is zero.*

## The Principle of Virtual Work.

**250.** The principle stated above, for the case of forces in equilibrium acting at a single point, is called *the principle of virtual work.* The conditions of equilibrium obtained in this case are identical with those obtained from the resolution of forces; for the work in any virtual displacement is merely the product of $\delta s$ and the sum of the resolved parts of the forces in the direction of displacement.

But we have seen (Art. 117) that a solid body acted upon by forces having several points of application may be regarded as a

system of interacting particles, each of which is in equilibrium under the action of certain forces. These forces taken together for all the points constitute the *external* and *internal* forces. Now, when the solid begins to move in any way, the several points of application will have certain initial velocities, which are sometimes called their *virtual velocities.* If we determine these virtual velocities, we can write the expression for the rate at which work begins to be done in the displacement; and, equating this to zero, we have a method of obtaining conditions of equilibrium distinct from the methods of resolving forces and of taking moments.

## Work done by Internal Forces.

**251.** The internal forces mentioned above are stresses between pairs of particles between which a mutual action exists. Let $A$ and $B$, Fig. 78, be two such points between which a stress $P$ exists, and let $P$ tend to *increase* the distance $AB$. Assume rectangular coordidates such that the axis of $x$ is parallel to $AB$, and let $x_1$ and $x_2$ be the abscissas of $A$ and $B$, so that $AB = x_2 - x_1$. Then the stress $P$ acts parallel to the axis of $x$ in the positive direction at $B$, and in the negative direction at $A$. Suppose now a displacement takes place in which $AB$ assumes the position $A'B'$. The virtual velocities of $A$ and $B$ are the projected velocities of these points *at the beginning of the motion* namely,

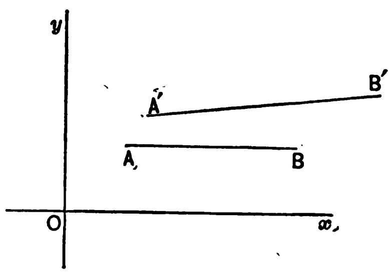


FIG. 78.

$$\frac{dx_1}{dt} \quad \text{and} \quad \frac{dx_2}{dt}.$$

Hence the rate at which $P$ does work at $B$ is $P\,\dfrac{dx_2}{dt}$, and that at

which work is done against $P$ at $A$ is $P\,\dfrac{dx_1}{dt}$. Thus the total work-rate of the two phases of the stress is

$$P\frac{d}{dt}(x_2 - x_1);$$

that is to say, it is *the product of the stress $P$ and the rate of change in the length of $AB$.* Denoting this length by $\rho$, the virtual work is $P\delta\rho$. This expression shows that the internal work of a system of interacting particles depends solely upon their *relative motions,* and not upon their absolute motions.

252. In the case of a rigid body the distance between any two points is invariable. Therefore the work of the stresses between the parts of a rigid body vanishes, so that *the internal forces do not appear in the expression for the virtual work in any displacement of the solid.* Thus, in the case of any forces acting in equilibrium upon a rigid body (but not at a single point of application), *the algebraic sum of the virtual works of the forces in any displacement is zero.*

### Virtual Work in Constrained Motion.

253. When a solid is free, a condition of equilibrium might be obtained from any virtual displacement which is possible to the body as a whole. The simplest displacements are translations in fixed directions, in which every point of application of an external force has the same displacement $\delta s$, and the condition of equilibrium is (as we have seen in Art. 250, in the case of a single point of application,) identical with that obtained by the resolution of forces.

254. The displacements next in point of simplicity are rotations about fixed axes. If we draw a perpendicular $AR$ from the point of application $A$ of a force to the axis of rotation, and resolve the force into three rectangular components, two of which are paralled to the axis and to $AR$ respectively, it is evident that only the third component, which we may denote by $P$,

does work, when the body undergoes a virtual angular displacement $\delta\theta$ about the axis. Moreover, the virtual displacement of the point $A$ is $AR\delta\theta$, hence the virtual work of the force is $P.AR\delta\theta$. Now, by Art. 218, $P.AR$ is the moment of the given force about the axis, since $P$ is the only component which has a moment about the axis and $AR$ is its arm. Therefore, in rotation, *the work done by a force is the product of the moment of the force and the angular displacement.* Now, in the expression for the total work, the angular displacement $\delta\theta$ will occur as a common factor; hence the total virtual work in the case of a system of forces is the product of the resultant moment of the system and the virtual angular displacement.

It follows that the condition of equilibrium obtained by the principle of virtual work, in the case of rotation, is identical with that obtained by taking moments about the axis.

255. But, when the body is constrained in its possible motions, we can, by considering only such displacements as are possible under the constraint, obtain equations which are free from the forces of constraint. For, at each point of contact of the body with fixed bodies, the reaction which constitutes the constraining force is perpendicular to the displacement at that point; that is, it does no work and therefore does not appear in the equation.*

It is in this way that we can, as mentioned in Art. 240, obtain directly the conditions of equilibrium for constrained bodies.

## Expression of the Total Virtual Work in the Displacement of a Solid.

256. As stated in Art. 241, the number of these equations is $m$, the number of the body's degrees of freedom; and, when the question is one of finding the position of equilibrium, the unknown quantities are in fact the $m$ numerical elements determin-

* The constraints are here supposed *smooth*, for if there were frictional resistances, as in the illustration of Art. 248, their limiting values would involve the values of the normal resistances.

ing the position, sometimes called *the coordinates of position of the constrained body.*

In any displacement of the solid, it is necessary, in order to obtain an expression for the work done, to express the linear virtual displacements of the points of application of the several forces in terms of these coordinates of position.

**257.** For example, the end $A$ of a uniform rod of length $2a$, and weight $W$, Fig. 79, is constrained to move in the horizontal line $AC$, while the end $B$ is constrained to move in a vertical line intersecting $AC$ in $C$; the constraints being smooth. By means of a string attached to $A$ and passing over a pulley at $C$, the weight of a given body $P$ acts at $A$ in the direction $AC$; to find the position of equilibrium.

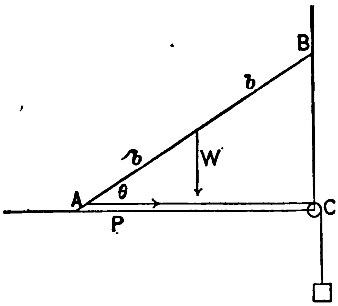


FIG. 79.

The rod has, in this case, but one degree of freedom, and accordingly its position is fixed by a single coordinate. Taking $\theta$, the inclination to the horizontal, for this coordinate, and denoting $AC$ by $x$, we have

$$x = 2b \cos \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The linear virtual displacement of the point of application $A$ in the direction of the force $P$ is $-dx$, because $P$ acts in the direction of $x$ decreasing. Hence the virtual work of $P$ is $-Pdx$. Differentiating equation (1), we have

$$dx = -2b \sin \theta \, d\theta;$$

therefore the virtual work of $P$, in the displacement of the solid indicated by $d\theta$, is $2bP \sin \theta \, d\theta$.

In like manner, denoting by $h$ the height of the point of application of $W$ above $AC$, the work of $W$ is $-Wdh$, and

$$h = b \sin \theta, \qquad dh = b \cos \theta d\theta.$$

Therefore the virtual work of $W$ is $-bW \cos \theta d\theta$, and the total virtual work is

$$(2bP \sin \theta - bW \cos \theta) d\theta.$$

Putting this equal to zero, we obtain for the position of equilibrium

$$\tan \theta = \frac{W}{2P}.$$

### Stability of Equilibrium.

**258.** When, as in the preceding example, the solid has but one degree of freedom, any point of it during its possible motion is describing a definite path. In a position which is not one of equilibrium, the point tends to move in one of the two opposite directions along this path. By applying at the point an additional force of proper magnitude in the direction opposite to that in which the point tends to move, the motion may be prevented, and the body put in equilibrium. If now the body be displaced in the direction of the previous tendency to motion, the whole virtual work of the forces (including the additional force) is zero; but the work of the additional force is negative, therefore the total work of the original forces is positive. It follows that, when a solid has one degree of freedom, *it tends to move in such a manner that the virtual work of the forces is positive.*

**259.** The principle proved above enables us to give a more complete discussion of the stability of equilibrium considered in Art. 188.

The rate at which the displacement or motion takes place is measured by the rate of change in the coordinate of position $\theta$, that is, $\frac{d\theta}{dt}$; and, if we divide the expression for the virtual work

by $dt$, we shall have the rate at which work is done by the forces, or *the work-rate* of the forces. The expression for the work-rate will then be of the form

$$f(\theta)\,\frac{d\theta}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

In general, $f(\theta)$ will have a finite value, and this expression will change sign with $\frac{d\theta}{dt}$; and we have seen in Art. 258 that the tendency to change in $\theta$ will be such as to give to expression (1) a positive value.

**260.** In a position of equilibrium, however, the initial work-rate is zero, therefore $\theta$ has a value, say $\theta_0$, which satisfies the equation $f(\theta) = 0$. Now, in order that the equilibrium may be stable, the body must tend to return to the position of equilibrium when displaced from that position, either by increasing or decreasing $\theta$ from the value $\theta_0$. Therefore the work-rate must in either case become *negative*. This requires that $f(\theta)$ shall become negative when $\theta$ increases and positive when $\theta$ decreases from the value $\theta_0$. In other words, $f(\theta)$ must *change from positive to negative as $\theta$ increases through the value $\theta_0$*. Now this will be the case if *the derivative of $f(\theta)$ is negative* when $\theta = \theta_0$, because this indicates that $f(\theta)$ is decreasing as it passes through zero, so that it takes values in the order $+$, $0$, $-$.

Thus, in the example of Art. 257, the value of $f(\theta)$ was

$$f(\theta) = 2bP \sin\theta - bW \cos\theta;$$

whence

$$f'(\theta) = -\,2bP \cos\theta - bW \sin\theta.$$

This expression is negative for $\theta_0$ the value of $\theta$ obtained for the position of equilibrium, therefore the position is one of *stable* equilibrium.

## Case of Several Degrees of Freedom.

**261.** When there are two degrees of freedom, any particular position of the solid is distinguished from other possible positions by particular values of two coordinates of position. Let these be denoted by $\theta$ and $\phi$. The linear virtual displacement of the point of application of any one of the forces due to any displacement of the solid may be expressed in terms of $d\theta$ and $d\phi$; hence the expression for the total virtual work will be in the form of two terms involving respectively $d\theta$ and $d\phi$. By equating to zero the coefficient of each of these terms separately, we have two equations each of which, in general, involves both $\theta$ and $\phi$. By the simultaneous solution of these equations we find, if possible, values $\theta_0$ and $\phi_0$ which determine a position of equilibrium.

**262.** The equations referred to above correspond to displacements of the solid in which one or other of the coordinates of position remains constant. In any other virtual displacement, $d\theta$ and $d\phi$ may be regarded as having some arbitrary ratio.

In order that the equilibrium in the position determined by $\theta_0$ and $\phi_0$ should be completely stable, the work-rate must become negative as soon as we make any displacement from the position $\theta_0$, $\phi_0$, with any values whatever of the rates $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$. We shall see hereafter that the criterion for stability is the same as that for a maximum value of a certain function of the two variables. See Art. 282.

The considerations adduced in this and the preceding article evidently extend to the case of three or more degrees of freedom.

## Determination of Unknown Forces by the Principle of Virtual Work.

**263.** The principle of virtual work may also be employed in problems where the unknown quantities required are not coordinates of position.

For instance, in the example of Art. 107, no motion of the beam is possible. The tension of the cord $AC$, Fig. 39, is as much a restraint as either of the other resistances $S$ and $R$. But, if we required only the value of $T$, we might substitute for it a force capable of doing work, while the resistance of the horizontal plane at $A$ and the top of the post at $D$ remain as forces of constraint. The beam is thus imagined to admit of a motion in which there is one degree of freedom. We may take as the coordinate of position, in this supposed motion of the beam, the length $AC$, which is now regarded as variable. Denoting it by $x$, and by $h$ the height of $M$ above the plane, we have by similar triangles, since $AM = 3$ and $DC = 3$,

$$\frac{h}{3} = \frac{3}{\sqrt{(x^2+9)}}; \qquad \text{whence} \qquad h = 9(x^2+9)^{-\frac{1}{2}}.$$

The virtual work of $T$ in the displacement determined by $dx$ is $-Tdx$, and that of $W$ is $-Wdh$. Differentiating the value of $h$, and substituting, the total virtual work is

$$-Tdx + 9W(x^2+9)^{-\frac{3}{2}}x\,dx.$$

Equating the coefficient of $dx$ to zero, we have for equilibrium a general relation between $T$ and $x$, namely,

$$T = 9x(x^2+9)^{-\frac{3}{2}}W;$$

and this, for the special value of $x$ in the problem, namely, $x = 4$, gives $T = \frac{36}{125}W$.*

* In this example, as in every case of plane motion, the motion begins to be the same as that of rotation about the instantaneous centre, which is in this case the intersection of the lines of action of $R$ and $S$. Hence, in accordance with Art. 254, the condition of equilibrium obtained is the same which would result from taking moments about that point. So also, in finding $S$ by the principle of work, the motion imagined would be that of free turning about $A$; therefore the relation between $S$ and $W$ would be the same as that derived in Art. 108 by the principle of moments.

## Equilibrium of Interacting Solids.

**264.** In the case of a system of interacting solids capable of restricted relative motions, the configuration (see Art. 138) is determined by one or more coordinates of relative position. Thus a rhombus $ABCD$, Fig. 80, formed of four equal jointed bars, capable only of plane motion, has its configuration determined by the value of the angle $\theta$ between the diagonal $DB$ and the side $AB$. If there are no external forces acting, the configuration of equilibrium (but not the absolute position of the system) may be determined by an equation of work involving internal forces which are capable of doing work. Thus, suppose the points $A$ and $C$ to be connected by an elastic string of natural length $l$ and strength $K$; that is to say, such that a force $K$ will produce an elongation of one unit of length in excess of its natural length $l$. By *Hooke's Law* the tension of such a string is proportional to its extension in length; therefore, when the string has the length $y$ its tension is $P = K(y - l)$. Again, if $B$ and $D$ are connected by an elastic string of natural length $l'$ and the *same* strength $K$, the tension of $BD$ when its length is $z$ is $Q = K(z - l')$. Now, in any displacement the total virtual work of the forces is $-Pdy - Qdz$, or

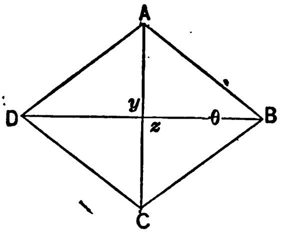

FIG. 80.

$$- K[(y - l)dy + (z - l')dz].$$

Denoting the length of each bar by $a$, the values of $y$ and $z$ in terms of $\theta$ are

$$y = 2a \sin \theta, \qquad z = 2a \cos \theta;$$

whence

$$dy = 2a \cos \theta \, d\theta, \qquad dz = - 2a \sin \theta \, d\theta.$$

Substituting, and equating the virtual work to zero,

$$2a \cos\theta \sin\theta - l\cos\theta - 2a\sin\theta\cos\theta + l'\sin\theta = 0;$$

from which we have

$$\tan\theta = \frac{l}{l'}$$

to determine the configuration of equilibrium. It is readily seen that the configuration is one of stable equilibrium.

## EXAMPLES. XIII.

1. A continuous source of energy which can do 33,000 foot-pounds of work in one minute, or 550 in one second, is said to have one *horse-power*. What is the horse-power developed by a locomotive which keeps a train weighing 50 tons moving uniformly at the rate of 30 miles an hour on a level track, the resistance being 16 pounds per ton? 64.

2. What should be the horse-power of a locomotive to move a train of 60 tons at the rate of 20 miles an hour up an incline of 1 foot in 100, the resistance from friction being 12 pounds per ton? 110.08.

3. Assuming a cubic foot of water to weigh $62\frac{1}{2}$ pounds, how many cubic feet will an engine of 25 horse-power raise per minute from a depth of 600 feet? 22.

4. A train, whose weight is 100 tons, is ascending uniformly a grade of 1 in 150, and the resistance from friction is 12 pounds per ton. If the locomotive is developing 200 horse-power, what is the rate in miles per hour? 27.85.

5. Determine the value of $\theta$ in the problem of Art. 81 by the principle of virtual work, and show that the position is one of unstable equilibrium. What may be inferred with respect to the locus of the centre of gravity of the weights $P$ and $Q$?

6. Solve Ex. VII, 2, by the principle of virtual work, and show that the position is one of unstable equilibrium.

7. The ends $A$, $B$ of a uniform heavy rod lie in a smooth ellipse whose major axis is vertical; referring the ellipse to the focus and directrix, prove that the rod is in equilibrium if it

passes through the lower focus. Show geometrically that this is a position of stable equilibrium.

8. Solve the problem of Art. 114 by virtual work, and prove that the equilibrium is unstable. What may be inferred with respect to the locus of the point $C$?

## XIV.

### Total Work of a Force in an Actual Displacement.

**265.** When the displacement is in the direction of the force, the element of work is $Pds$; and, if this direction remains constant while the force $P$ is variable, the whole work done in a displacement is the integral

$$\int Pds$$

taken between proper limits, where $P$ is supposed to be expressed as a function of $s$.

As an illustration, consider the force of an elastic string or wire $AB$ which is stretched beyond its natural length $l$ to a length $l + s_1$. Suppose the end $A$ to be fixed and the force which produces the extension $s$ in the length to act at $B$ in the direction $AB$. By *Hooke's Law*, the tension of the stretched string, when it has any extension $s$ beyond its natural length, is directly proportional to the extension, and may be denoted, as in Art. 264, by $Ks$, where $K$ is a constant which is called the strength of the spring, because it is the value of the force when the extension is unity. Treating $s$ as a variable, $P = Ks$ is the expression for the variable force producing the extension in terms of $s$, which is measured from $B$, so that the lower limit of the integral expressing the work is zero. Thus the total work done in producing the extension $s_1$ is

$$\int_0^{s_1} Pds = K\int_0^{s_1} sds = \tfrac{1}{2}Ks_1^2.$$

Since $Ks_1$ is the final value of the variable force, it appears that, when Hooke's Law applies, the work done is one-half as much as would have been done had the force been constant and equal to its final or greatest value.

## Graphical Representation of Work.

**266.** If, at all points of the line in which the point of application of a variable force travels, perpendiculars be erected, representing on some selected scale the corresponding values of the force, the curve marked out by their extremities will give a graphic representation of the mode in which the force varies with the space. This is, for example, mechanically done in the formation of the "indicator diagram" of a steam-engine. Thus, in Fig. 81, suppose the horizontal motion of the pencil describing the curve $A_1A_2$ to represent (generally on a reduced scale) the motion of the piston, while the distance $AB$ of the pencil from the horizontal line $Ox$ is by proper mechanism caused to be always proportional to the force acting on the piston. Taking as origin the point from which $s$ is measured, we thus have a curve in which the abscissa and ordinate of any point represent corresponding values of $s$ and $P$.

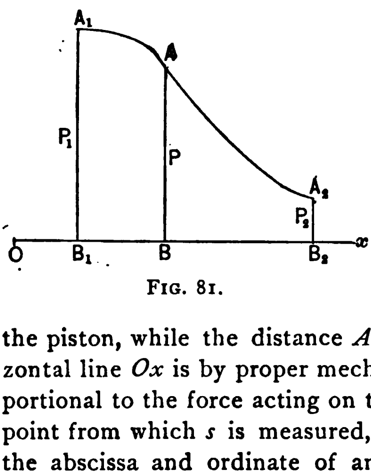


FIG. 81.

**267.** Let $A_1A_2$, Fig. 81, be any curve of force thus constructed: the integral $\int_{s_1}^{s_2} Pds$, which represents the work done when the point of application passes over the space $B_1B_2 = s_2 - s_1$, is also the value of the area $A_1A_2B_2B_1$ inclosed between the curve, the axis of abscissas and the ordinates corresponding to the limits. Hence, by the construction of the curve of force, we are able to represent the work done by an area. When the

curve is mechanically constructed as supposed in Art. 266, the area is measured either by one of the approximate methods or by the Planimeter.

**268.** The graphic representation of force by an area is sometimes useful when the law of the force is known. In this case, the curve of force will be a known curve, and the areas representing the work may be obtained directly from geometrical principles. For example, in the illustration of Art. 265 let $AB$, Fig. 82, be the natural length of the string, and $BC = s_1$ the final extension. Hooke's Law, $P = Ks$, gives for the curve of force the straight line $BD$ passing through $B$, the origin from which $s$ is measured. The value of $CD$ is the final value of the force, $Ks_1$, and the area of the triangle $BCD$, which represents the work done, is one-half the product of the base and altitude, that is, $\frac{1}{2}Ks_1^2$. In like manner the work done in any displacement not starting from the point of zero force $B$ would be represented by a trapezoid, and its value found from the known expression for the area of a trapezoid.

FIG. 82.

## Work done when the Path of Displacement is Oblique or Curved.

**269.** When the elementary displacement $ds$ makes the angle $\phi$ with the direction of the force, the element of work is $Pds \cos \phi$. In this expression, $\phi$ may be variable either on account of a change in the direction of the force or of that of the path described by the point of application. The total work is now the value between proper limits of the integral

$$\int P \cos \phi ds,$$

in which $P$, $\phi$ and $s$ are supposed to be expressed in terms of a single variable.

**270.** Let us consider first the special case in which the force $P$ is the weight $W$ of a particle, which is constant in amount and direction, while the path described is curved so that $\phi$ is variable. Referring the curved path, Fig. 83, to rectangular coordinates, the axis of $x$ being horizontal, $\phi$ is the inclination of the curve to the axis of $y$ so that

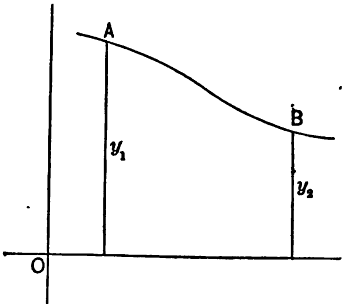

FIG. 83.

$$\cos \phi = \frac{dy}{ds}.$$

Making this substitution, and putting $P = - W$ because the direction of the force is that in which $y$ is negative, the expression for the work done by gravity is

$$\int P \cos \phi \, ds = - W \int_{y_1}^{y_2} dy = W(y_1 - y_2);$$

where $y_1$ and $y_2$ are the initial and final height of the particle above the axis of $x$. In the diagram, the work done is positive because $y_1$ is represented as greater than $y_2$.

The result shows that, in this case, the work done depends only upon the initial and final positions of the point of application, being independent of the path by which it passes from one position to the other.

## Potential Energy.

**271.** The body, in Fig. 83, where $y_1 > y_2$, is said to have a greater *potential energy* when at the point $A$ than when at the point $B$. The difference of potential energy at these two points, that is, the potential energy expended in the displacement of the body, is taken as equivalent to the work done, $W(y_1 - y_2)$. In the reverse displacement, the work done against the force is, in

like manner, represented by an equivalent gain in potential energy.

When it is desired to assign an absolute value to the potential energy at a point, it is necessary to assume the position of *zero-potential*. For example, in the present case, it is convenient to assume the potential to be zero on the axis of $x$, so that $Wy_1$ is the potential at $A$, and $Wy_2$ that at $B$. The value of the potential when the body is below the axis of $x$ is negative, but there is still a positive expenditure of potential energy when it passes to a position for which the value of the potential is algebraically smaller.

272. It is not necessary to suppose the curve in Fig. 83 to be a plane curve. The locus in space of the points of zero-potential is, of course, a horizontal plane. Accordingly the potential has a common value for all points upon any other given horizontal plane. For this reason, such planes are called, with reference to gravity, *equipotential surfaces*. In passing from one such surface to another, the loss of potential is equal to the work done by the force, or the gain of potential is the work done against the force. If the path begins and ends in the same potential surface, that is, *at the same level*, there is as much work done by gravity as against it, and the total work is zero.

The force of gravity is called a *conservative force*, because the work done against it is stored up, as it were, in the form of potential energy, or difference of potential, and can be reconverted into an equivalent amount of work. On the other hand, a force like that of friction is *non-conservative*, because work done against it does not produce any potential energy or power to do work.

## Work done by a Resultant.

273. We have seen in Art. 247 that, when constant forces are acting at a single point of application which undergoes displacement, the algebraic sum of the works done by the several forces has the same value as the single expression for the work done by the resultant. The same thing is obviously true of the ele-

ments of work done in the elementary displacement $ds$, when the forces are variable. Thus, using the same notation as in Art. 247,

$$R \cos \phi \, ds = P_1 \cos \phi_1 ds + P_2 \cos \phi_2 ds + \ldots + P_n \cos \phi_n ds.$$

Integrating between the same limits throughout for $s$, we have

$$\int R \cos \phi \, ds = \int P_1 \cos \phi_1 ds + \int P_2 \cos \phi_2 ds + \ldots + \int P_n \cos \phi_n ds,$$

which shows that the same principle applies to the total work done by variable forces in any displacement.

**274.** We cannot apply a like principle to the case of forces with different points of application, because we have, in the general case, defined only the line of action of the resultant and *not* any definite point of application. But, in the important special case where the forces are the weights of bodies (or of the parts of a solid body), a special point of application of the resultant, namely, the centre of gravity, has been exactly defined. In this case, we can show as follows that the total work done may be regarded as done by the resultant:

Let the bodies be referred to rectangular coordinate planes, as in Art. 178; then the height $\bar{z}$ of the centre of gravity above the horizontal plane of $xy$ is defined by the equation

$$\bar{z} \Sigma P = \Sigma z P.$$

Now by Art. 271 the second member of this equation is the total potential of the weights, when the plane of $xy$ is taken as that of zero-potential. In like manner, the first member is the potential of the total weight regarded as situated at the centre of gravity. Hence the equation may be regarded as expressing the equality of these potentials; and, since the work done in passing from one configuration of the bodies to another is equal to the loss of potential, the total work done is equal to that which would be done by the total weight at the centre of gravity.

The result of course applies to a continuous body of any

form; thus, for example, the work of emptying a cistern of water by a pump is equal to that of raising a weight equal to that of the water through the vertical height of the point at which the water is discharged above the centre of gravity.

### Work expressed in Rectangular Coordinates.

**275.** When the point of application of a variable force $P$ acting always in one plane is referred to *rectangular* coordinates, $ds$ denotes an element of the path described by the point of application $(x, y)$, and $dx$ and $dy$ are the projections of $ds$ in the directions of the axes of $x$ and $y$ respectively. Then $X$ and $Y$ being the rectangular components of $P$, $Xdx$ is the work done by $X$, and $Ydy$ that done by $Y$ in the displacement $ds$. Therefore, since by Art. 273 the work done by the components is equal to that done by the resultant, the work done by $P$ is

$$Xdx + Ydy,$$

a result which is readily verified by substituting values in terms of $P$, $ds$ and their inclinations to the axis of $x$.

**276.** In like manner, in the general case, using three rectangular planes of reference, we have for the work of $P$ when the displacement is $ds$

$$Xdx + Ydy + Zdz,$$

which can be verified by the substitution of values in terms of $P$ and its direction angles $\alpha$, $\beta$, $\gamma$ for the forces, and of $ds$ and its direction angles, $\lambda$, $\mu$, $\nu$ for the displacements.

The expressions for work just derived show that the condition found in Arts. 77 and 80, for equilibrium upon a fixed curve and upon a fixed surface, express the fact that the virtual work is zero when the body is displaced along the curve, or in any direction along the surface, in accordance with the principles established in the preceding section.

## The Work Function for Central Forces.

277. A central force is a variable force which acts upon a particle in such a manner that the line of action always passes through a fixed point called *the centre of force*, and that the magnitude of the force is expressible as a function of the distance of the particle from the centre of force. Denoting this distance by $r$, we have then

$$P = f(r),$$

where $f$ denotes some given function. The intensity of a central force is thus the same for all points situated upon the surface of a sphere whose centre is at the centre of force, but, in general, differs for the surfaces of two such concentric spheres.

Now, if the particle undergoes any elementary displacement $ds$, the displacement in the direction of the force is $dr$, which is the projection of $ds$ upon the line of action $r$. Hence the work of the force in the displacement $ds$ is

$$Pdr = f(r)dr.$$

This expression is positive when the direction of the force $P$ is that of $r$ increasing, that is, when the force is *repulsive*. In this case, work is done by the force when $r$ increases. On the other hand, if $P$ is an *attractive* force, $f(r)$ is negative, and work is done against the force when $dr$ is positive or $r$ increasing.

278. The function of which $Pdr$ is the differential is called *the work-function* for the force. Denoting it by $V$, we have

$$dV = Pdr = f(r)dr, \qquad \text{or} \qquad V = \int f(r)dr + C.$$

In this indefinite form of the integral there is an arbitrary constant which is generally so taken that $V$ vanishes for a certain initial value $r_0$. The work done in a displacement from any initial distance

$r_1$ to the distance $r_2$ is the integral between these limits of the elementary work $Pdr$. Hence it is

$$\int_{r_1}^{r_2} Pdr = V_2 - V_1.$$

If the lower limit is $r_0$ for which $V$ vanishes, the work done in displacement to any distance $r$ is simply $V$.

## The Potential Function.

**279.** The function whose derivative is the negative of that of the work function is called *the potential function*. Denoting it by $U$, we have

$$U = C - V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $C$ is a constant. Supposing, as in the preceding article, that $V$ is so taken as to vanish when $r = r_0$, the corresponding value of $U$ is $U_0 = C$. For any value of $r$, $V$ is the work done by the force in a displacement from the initial distance $r_0$ to the distance $r$, and $C$ is the value of the potential energy when $r = r_0$. Therefore $U$ is the value of the potential energy left after the force has done the work $V$. When work is done *against* the force in displacing the particle from the distance $r_0$ to the distance $r$, $V$ is negative and $U$ takes a value greater than its initial value $C$. Putting the equation in the form

$$U + V = C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it expresses that the sum of the work done by the force and the potential energy (or remaining power to work) is constant. A central force is therefore, like the force of gravity (see Art. 272), said to be a conservative force; work done by it is said to be due to the expenditure of potential energy and work done against it is stored up in the form of potential energy.

**280.** Since, in the present case of a central force, $U$ is a function of $r$, it has the same value for all points at the same distance from the centre of force. Thus the *equipotential surfaces* are con-

centric spherical surfaces each characterized by a special value of the potential. In passing from one of these surfaces to another the force does an amount of work which is *independent of the path of displacement*, and is equal to the difference of potential of the extreme points. The loss of potential in any displacement is exactly equal to the gain of potential which would result from the reverse displacement.

## The Potential of Attractive Force Varying directly as the Distance.

**281.** As a single illustration of the potential of a central force, let us suppose $P$ to be an attractive force varying directly with the distance $r$, so that we may put

$$P = -\mu r,$$

where $\mu$ is a positive number expressing the intensity of the force at a unit's distance. In this case, the work function, Art. 278, is

$$V = \int P dr = -\mu \int r dr = -\tfrac{1}{2}\mu r^2,$$

in which the constant of integration has been so taken that $V$ vanishes for the initial value $r_0 = 0$. The negative value of $V$ indicates that work must be done against the force in displacing the particle from the centre of force to any distance $r$ from it.

It is convenient in this case to assume $C = 0$ in the equations of Art. 279. Thus the potential function is

$$U = \tfrac{1}{2}\mu r^2.$$

Its value at any distance $r$ is therefore the work which would be done in removing the particle from the centre of force to a point at the distance $r$ from it.

In taking the constant as we have done we have assumed the centre to be the point of zero-potential.

## The Work Function in General.

**282.** The positions of the bodies of a system depend upon the values of a number of determining quantities or coordinates. Now the forces which act upon the bodies are either stresses acting between them or between them and external bodies, although the latter are generally treated as if directed to fixed centres of force. The intensities of the forces are in all actual cases functions of the distances between the bodies upon which they act, so that for each force the element of work done is an expression involving one of these distances and its differential. Such an expression, being a function of a single variable, is an exact differential expression, and therefore the total differential expression for the work is an exact differential; that is to say, the differential of some function of the several variable distances.

Now this will still be true when the total element of work is expressed in terms of the coordinates, such as $\theta$ and $\phi$ in Art. 262, which determine the position of the system which may be subject to any given constraints. We therefore conclude that, in all cases of actual forces, there exists a function of the coordinates of position, of which the differential expresses the element of work done in any possible displacement. This function is called the Work-Function; it increases whenever positive work is done; that is, for every displacement which *tends* to take place. (Compare Art. 258.) Therefore it has a maximum value at a position of stable, and a minimum, at one of unstable equilibrium.

**283.** We shall therefore assume that the work done in an elementary displacement $ds$ of the free particle acted upon by the variable force $P$, namely, (Art. 276,)

$$Xdx + Ydy + Zdz,$$

is the exact differential of some function $V$ of $x, y$ and $z$; that is, we assume

$$dV = Xdx + Ydy + Zdz. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$V$ is called the work-function, and the expressions for the rectangular component forces in terms of $V$ are

$$X = \frac{dV}{dx}, \qquad Y = \frac{dV}{dy}, \qquad Z = \frac{dV}{dz}, \quad . \quad . \quad . \quad . \quad (2)$$

the partial derivatives of $V$ with respect to $x, y$ and $z$ respectively. It must be remembered that in accordance with the notation of the Differential Calculus $dV$ has a meaning in each of the fractions different from that of $dV$ in equation (1). For this reason we shall here use the notation of the virtual displacements of the preceding section, and write equation (1) in the form

$$\delta V = X\delta x + Y\delta y + Z\delta z, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in which $\delta x$, $\delta y$ and $\delta z$ are the projections in the direction of the axes of the virtual displacement $\delta s$, and accordingly $\delta V$ is the virtual work done in the displacement $\delta s$.

284. Dividing this equation by $\delta s$, we have

$$\frac{\delta V}{\delta s} = X\frac{\delta x}{\delta s} + Y\frac{\delta y}{\delta s} + Z\frac{\delta z}{\delta s} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, denoting the direction angles of the displacement $\delta s$ by $\lambda$, $\mu$, $\nu$, this becomes

$$\frac{\delta V}{\delta s} = X\cos\lambda + Y\cos\mu + Z\cos\nu. \quad . \quad . \quad . \quad (5)$$

Since $\delta V = P\cos\phi\,\delta s$, the second member of this equation is an expression for the resolved force in the direction of the displacement.*

* This is readily seen geometrically, for if $P$, $X$, $Y$ and $Z$ are constructed as in Fig. 17, p. 45, $X\cos\lambda$ is the projection of $X$ upon a straight line in the direction of $\delta s$, $Y\cos\mu$ that of $Y$, and $Z\cos\nu$ that of $Z$. Their sum is therefore the projection of $P$ on the same line, that is, $P\cos\phi$.

## Equipotential Surfaces.

**285.** The potential function $U$ is defined as in Art. 279, so that $U = C - V$, and its derivates are simply those of the work-function with their signs changed. If we put $U = C$, where $C$ is any constant we have the equation in $x$, $y$, $z$ of a surface. If the displacement $\delta s$ takes place in any direction along this surface, we have $\delta U = 0$, that is to say, no work is done. Such a surface is called an *equipotential surface*. We have already seen that, for a constant force such as gravity, these surfaces are parallel planes; also that, for any central force, they are concentric spherical surfaces. In general, they form a system of surfaces which do not intersect one another; for $U = C$ and $U = C'$ are contradictory equations.

The direction of the force is at every point of an equipotential surface normal to it.* A line (in the general case a curved one) which is normal to every surface of the system is called a *line of force*.

**286.** By integration of the element of work, equation (1),

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* If $\alpha$, $\beta$, $\gamma$ are the direction-angles of the force $P$,

$$X = P\cos\alpha, \qquad Y = P\cos\beta, \qquad Z = P\cos\gamma.$$

Accordingly the partial derivatives of $V$, see Art. 283, are proportional to the direction-cosines of the normal to the surface $V = C$.

When these values are substituted in the expression for the derivative of $V$ in the direction of $\delta s$ (of which the direction-angles are $\lambda$, $\mu$, $\nu$), equation (3), Art. 283, we have

$$\frac{\delta V}{\delta s} = P(\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu) = P\cos\phi.$$

This derivative may be called *the space-rate of energy expended;* it is zero for any direction $\lambda$, $\mu$, $\nu$ which satisfies

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 0,$$

that is, for any tangent line to the surface, and it is a maximum when

$$\cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu = 1,$$

which is satisfied *only* by $\lambda = \alpha$, $\mu = \beta$, $\nu = \gamma$.

Art. 283, we find that the work done in any displacement from the point $A$ to the point $B$ is

$$V_2 - V_1,$$

where $V_2$ is the value of the work-function at the point $B$, and $V_1$ that at $A$. If we use instead the potential function, it is

$$U_1 - U_2;$$

that is, the work done is equal to the loss of potential, and negative work is represented by gain of potential, as in the special cases already considered.

287. If the particle is so constrained that it can move only in a path which crosses the equipotential surfaces, it will tend to move from the surface of higher to that of lower potential; that is, in accordance with Art. 258, in that direction in which the virtual work of the forces is positive. At a point where the path is tangent to an equipotential surface no virtual work is done, and we have a position of equilibrium. Supposing the path to suffer no sudden changes of direction, a position of maximum or minimum potential is such a point, the equilibrium being unstable in the first case and stable in the second. If the path were tangent to an equipotential surface but also crossed it, the equilibrium would be stable on one side, and unstable on the other side, of the position of equilibrium.

In like manner, when the particle is restricted only to remain in a given surface, a position of maximum potential is one of unstable equilibrium, and one of minimum potential is one of stable equilibrium.*

* The intersections of the surface of a mountain with horizontal planes at different altitudes, which are the equipotential surfaces in the case of gravity are called *contour lines*. These lines form a good illustration in two-dimensional space of the equipotential surface in three-dimensional space; the variable force to which they correspond being the component of gravity along the sloping surface. As we pass from one contour line to another, the work done is the difference of corre-

## EXAMPLES. XIV.

1. What is the work done in raising to the surface the water in a cistern 10 feet square and 6 feet deep? 112,500 ft.-lbs.

2. A well is to be made 20 feet deep and 4 feet in diameter. Find the work in raising the material, supposing that a cubic foot of it weighs 140 lbs. 351,900 ft.-lbs.

3. What part of the work of emptying a conical cistern is done when the depth is reduced one-half? $\frac{11}{16}$.

4. Find how many units of work are stored up in a mill-pond which is 100 feet long, 50 feet broad, and 3 feet deep, the point at which the water is discharged being 11 feet below the surface of the pond. 8,906,000.

5. Show that, in accordance with Hooke's Law, the work done in stretching the string through any space is the product of the space and the arithmetical mean of the initial and final tensions.

6. The wire for moving a distant signal is, when the signal is down, stretched 16 inches beyond its natural length, and has a tension of 240 pounds, which is produced by a back weight of 270 pounds resting with a portion (30 pounds) of its weight upon its bed. If the signal end of the wire is to move through 2 inches in raising the signal, show that the end which is attached to the hand-lever must have a motion of 4 inches. Find the work done when the hand-lever is suddenly pulled back and locked before the signal begins to move, and find how much less work is necessary if it be pulled back slowly. 90 ft.-lbs.; $2\frac{1}{2}$ ft.-lbs.

7. Assuming the earth to be a sphere of radius $a$, and the at-

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sponding potentials. The lines of greatest slope are those which at every point give the direction of the greatest force, thus corresponding to the "lines of force" of Art. 285. Accordingly, they cross the contour lines at right angles, just as the lines of force cross the equipotential surfaces at right angles. In passing between consecutive contour lines along a path oblique to the line of greatest slope, the distance is increased in the same ratio as that in which the effective force or force along the path is diminished, so that the product, or work done, is unchanged.

traction of a body to the earth to be inversely proportional to the square of its distance from the centre, show that the work done in removing a body whose weight is $W$ from the surface to an infinite distance is $Wa$.

8. Show that the work done against friction in dragging a body along a rough curve in a vertical plane, by a force which is always tangent to the path, is independent of the form of the curve.

9. Show that, if the force in Ex. 8 makes a constant angle $\beta$ with the tangent to the path, but never becomes vertical, the whole work done is still independent of the form of the path, and find its ratio to that done when $\beta = 0$, $\alpha$ being the angle of friction. $\dfrac{\cos\beta\cos\alpha}{\cos(\beta-\alpha)}$.

10. A weight $W$ moving in vertical guides rests upon a bar which turns upon a horizontal axis at the distance $a$ from the guides. The weight is raised by turning the rod through the angle $\theta$ from the horizontal position. Show that, if $\mu$ and $\mu'$, the coefficients of friction between the weight and guides and the weight and rod respectively, are small, the work done against friction is approximately

$$\tfrac{1}{2}Wa(\mu + \mu')\tan^2\theta.$$

11. A weight $W$ is drawn up a rough conical hill of height $h$ and slope $\alpha$, and the path cuts all the lines of greatest slope at the constant angle $\beta$. Find the work done in attaining the summit. $Wh(1 + \mu\cot\alpha\sec\beta)$.

12. In the example of Art. 78, p. 56, show that the equipotential lines are circles, and that at positions of equilibrium not on the axis (when they exist) the equilibrium is stable.

13. A weight $W$ attached by a string to a ring moving on a smooth horizontal rod hangs vertically, the string passing through a fixed smooth ring at a distance $b$ below the rod. Verify, by direct integration of the work done in removing the ring through a distance $s$, that it is the same as that of raising the weight. Find also the work done against friction if the rod be rough.

$$\mu bW[\log\{\sqrt{(b^2 + s^2)} + s\} - \log b].$$

## CHAPTER VIII.

## MOTION PRODUCED BY CONSTANT FORCE.

### XV.

### Inertia regarded as a Force.

**288.** We have seen in Art. 13 that the property of matter through which it resists any change of motion, in accordance with the First Law of Motion, is called Inertia. The change of motion which is resisted is measured by the product of the mass and acceleration, that is, by $m\alpha$, which, in accordance with the Second Law, is taken as the measure of the force which, acting freely, produces the motion. Now, just as the resistance of a fixed body in contact with that upon which the force acts, and preventing its motion, is regarded as a force equal and opposite to the force which would otherwise produce motion, so the resistance to motion in the body when free is regarded as a force equal and opposite to the active force which produces the motion. Thus *the force of inertia* acts upon a particle of mass $m$ only when there is an acceleration $\alpha$, and its value is $m\alpha$, while its direction is opposite to that of the acceleration.

### The Centre of Inertia.

**289.** When a rigid body has a motion of translation (see Art. 1), all its points have at every instant a common velocity, and therefore a common acceleration; so that the forces of inertia acting on its several parts form a system of parallel forces pro-

portional to the masses of the parts, exactly as the forces of gravity do. It follows that, in this case, the resultant of the inertia forces is their sum acting at the same point as the resultant of the gravity forces regarded as a system of parallel forces. This point, usually known as the Centre of Gravity, is in fact more properly called *the Centre of Inertia.* Thus, for motions of translation, a rigid body may be regarded as a particle situated at the Centre of Inertia; and the weight of the body, when that is in question, is a force acting at the same point.*

## Rectilinear Motion.

**290.** We now resume that part of Dynamics to which Chapter I is introductory, namely, that which treats of the action of forces in overcoming the resistances of inertia. It is known as Kinetics,† because it is concerned with the production of motion.

We consider in this chapter the motion produced in a particle (or a solid regarded as a particle of mass $m$ situated at its Centre of Inertia) by a force constant in direction and magnitude; and, in the present section, we further suppose the particle to have no motion except in the line of action of the force.

In Art. 17, it is pointed out that the units of force, mass and acceleration are so taken that $F = mf$, where $f$ stands for the acceleration produced by $F$ acting freely; but, in virtue of this equation, $f$ may also be taken as the force acting upon a unit of

* The position of the Centre of Inertia of a body depends only upon its volume and the distribution of its mass. The identity of the Centre of Gravity with this point is due to the fact that we regard the forces of gravity, near the earth's surface, as constant forces proportional to the masses and acting in parallel lines.

† From κίνησις, movement. The term Dynamics, from δύναμις, force or power, is sometimes used to cover the whole range of Theoretical Mechanics. It has in this book been employed, in accordance with common usage (in such phrases, for example, as "dynamical friction"), to imply the action of a force through a space. Compare Art. 244.

mass. When only a single body is in consideration, we may omit the factor $m$, and equate the force $f$ acting upon the unit mass to the acceleration produced. For rectilinear motion in the line of action of the force, the acceleration is

$$\alpha = \frac{dv}{dt} = \frac{d^2s}{dt^2},$$

where $v$ denotes the speed, and $s$ the distance of the particle from some fixed origin taken on the line of motion. The differential equation

$$\frac{d^2s}{dt^2} = f,$$

where $f$ is the "accelerating force," or force acting on each unit of mass, is called *the equation of motion* for a particle moving in a straight line.

**291.** The solution of this equation of the second order is the relation between the variables $s$ and $t$ found by integration, and involving two constants of integration. But, since in Mechanics the velocity defined by the equation

$$v = \frac{ds}{dt}$$

is a variable of equal importance with $s$, we may with advantage regard this equation together with the equation of motion in the form

$$\frac{dv}{dt} = f$$

as two simultaneous differential equations of the first order between the three variables $v$, $s$ and $t$.

For integration, these equations are written in the form

$$dv = f\,dt, \quad \ldots \quad (1)$$

$$ds = v\,dt. \quad \ldots \quad (2)$$

Eliminating $dt$ between them, we also have

$$vdv = fds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

a differential relation between $v$ and $s$.

## Integration of the Equation of Motion when the Force is Constant.

**292.** When $f$ is a constant, equation (1) contains only two variables and can be directly integrated. The result may be written

$$v = v_0 + ft, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the constant of integration is expressed by the symbol $v_0$, because it is the value of $v$ when $t = 0$.

Using this value of $v$, equation (2) becomes

$$ds = v_0 dt + ft\,dt.$$

Hence a second integration gives

$$s = s_0 + v_0 t + \tfrac{1}{2}ft^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which the constant of integration is denoted by $s_0$ because it is the value of $s$ when $t = 0$.

This last equation is the complete solution of the differential equation of the second order,

$$\frac{d^2s}{dt^2} = f,$$

when $f$ is constant, and equation (1) is called a *first integral* of that equation.

**293.** Equation (3), Art. 291, contains only the variables $v$ and $s$; it therefore also admits of direct integration, giving

$$\tfrac{1}{2}v^2 = fs + C.$$

This is also a first integral of the differential equation of the second order. If we determine the constant of integration by means of the condition that $v = v_0$ when $s = s_0$ as in the preceding article, we shall have

$$\tfrac{1}{2}(v^2 - v_0^2) = f(s - s_0). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This relation between $v$ and $s$ might have been found by elimination of $t$ from equations (1) and (2); in other words, by eliminating $t$ *after*, instead of before, integrating.

## Kinetic Energy.

**294.** The equation

$$vdv = fds$$

is integrable, not only when $f$ is constant, but when it is a variable depending for its value only upon $s$; that is to say, when $f$ is a function of $s$. Multiplying by $m$, the mass of the body, and integrating, we have, since $F = mf$,

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = \int_{s_0}^{s} Fds,$$

where $v_0$ is the velocity corresponding to the lower limit $s_0$. The second member is, by Art. 265, the work done by the force $F$ in the displacement of its point of application (which is the particle, or the centre of inertia of the body) through the space $s - s_0$. The quantity $\tfrac{1}{2}mv^2$ is known as *the kinetic energy* of the mass $m$ moving with the velocity $v$. If $v_0 = 0$, the equation expresses that the kinetic energy is equal to the work done by the force $F$ in imparting to the body the velocity $v$. Thus kinetic energy is the measure of the work done by a force against inertia; and the general equation asserts that the work done by a force upon a body moving in the line of action is equal to the gain in kinetic energy. If the force is opposite in direction to the displacement, the work is negative, and there is a loss of

kinetic energy, which is thus equal to the work done by inertia against the force.

The equation of this article is called *the equation of energy.*

## Laws of Falling Bodies.

**295.** In the particular case of a body falling freely from rest, the position of rest is usually taken as the origin of $s$, and the instant of falling as the origin of time, or instant when $t = 0$; thus the "initial circumstances" or known corresponding values of the variables are $t = 0$, $s = 0$, $v = 0$. The space being measured downward, that is, in the direction of the force, the acceleration is positive and its value is $g$. Hence

$$\frac{d^2s}{dt^2} = g.$$

Integrating successively with respect to $t$, and determining the constants by the initial circumstances, we have

$$v = gt, \qquad \qquad (1)$$

$$s = \tfrac{1}{2}gt^2. \qquad \qquad (2)$$

Eliminating $t$ between these equations, we have also

$$v^2 = 2gs. \qquad \qquad (3)$$

These three equations, expressing the relations between each pair of the variables $t$, $v$ and $s$, are sometimes said to express the laws of freely falling bodies. It must be remembered that in accordance with the initial circumstances $t$ denotes the time in which the velocity $v$ is acquired, and in which the space $s$ is described, from rest.

**296.** The second equation shows that the space fallen through in the first second is $\frac{1}{2}g$, which is one-half the space that repre-

sents the velocity acquired. Again, the space described in the interval between the instants $t_1$ and $t_2$ is

$$\tfrac{1}{2}g(t_2^2 - t_1^2) = g\frac{t_2 + t_1}{2}(t_2 - t_1).$$

Defining *the average velocity* in a given interval as that with which, as a constant velocity, the body would describe in the interval a space equal to that which actually is described, and denoting the average velocity in the interval $t_2 - t_1$ by $v_m$, the space described is $v_m(t_2 - t_1)$. Comparing this with the expression written above, we see that the average velocity in any interval is

$$v_m = g\,\frac{t_2 + t_1}{2} = \tfrac{1}{2}(v_2 + v_1),$$

where $v_1$ and $v_2$ are the velocities at the beginning and end of the interval. That is, the average velocity, in the case of constant acceleration, is the arithmetical mean of the extreme velocities: it is also the same as the velocity at the middle instant.

The average velocity during the $n$th second, found by putting $t_2 = n$ and $t_1 = n - 1$, is accordingly the same as the space described in that second, namely,

$$\tfrac{1}{2}(2n - 1)g.$$

Thus the spaces described in successive seconds are proportional to the successive odd numbers.

297. The velocity acquired by falling from rest through the height $h$ is, by equation (3),

$$v = \sqrt{(2gh)}$$

This velocity is often called *the velocity due to the height h*. Conversely, the height

$$h = \frac{v^2}{2g}$$

is called *the height due to the velocity* $v$. It is the distance through which gravity must work upon a body originally at rest to give it the velocity $v$ or the kinetic energy $\frac{1}{2}mv^2$. Accordingly, multiplying by $W$, we have $Wh = \frac{1}{2}mv^2$.

## Body Projected Upward.

298. In the case of a body projected upward, it is convenient to measure the space positively upward: therefore gravity produces a retardation of $g$ feet. Taking the point and the instant of projection as the origins of space and time, and $v_0$ as the velocity of projection, the initial circumstances are

$$v = v_0, \quad s = 0 \quad \text{when} \quad t = 0.$$

Integrating

$$\frac{d^2s}{dt^2} = -g$$

successively, and determining the constants accordingly, we find

$$v = v_0 - gt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = v_0t - \tfrac{1}{2}gt^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, eliminating $t$,

$$v^2 = v_0^2 - 2gs. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

299. Equation (1) shows that the velocity, originally positive, is decreasing; it vanishes when $t = \dfrac{v_0}{g}$, which is the time required for gravity to overcome the initial velocity. Equation (2), being a quadratic for $t$ when $s$ is given, shows that there are two instants at which $s$ has a given value. For example, if the velocity of projection is $64^{f}/_{s}$, to find the instant when the body is 48 feet above the point of projection, we have the quadratic

$$48 = 64t - 16t^2.$$

The roots of this are $t = 1$ and $t = 3$; the first indicates the

instant at which the body reaches the height of 48 feet while rising, and the second that at which it returns to the same point while falling.

**300.** Equation (3) shows that the two values of $v$ which correspond to the same value of $s$ are numerically equal and of opposite signs, that is, the body passes a given point with the same speed in ascending and descending.

The greatest height $H$ from the ground to which the body will rise is found by putting $v = 0$ in the same equation to be

$$H = \frac{v_0^2}{2g},$$

which is the height due to the initial velocity $v_0$ (see Art. 297). The height due to a given velocity may therefore be defined as that to which a body will rise if projected directly upward with that velocity.

**301.** Multiplying equation (3) by $\frac{1}{2}m$, and introducing $H$ in place of $v_0$, we derive the equation

$$\tfrac{1}{2}mv^2 = mg(H - s) = W(H - s),$$

which shows that the kinetic energy at any point is equal to the work of gravity corresponding to the distance of the point below the highest point reached.

If we take the ground, that is, the level of the point of projection, as that of zero-potential (see Art. 271), $Ws$ is the potential energy of the body when at the height $s$. Then, writing the equation in the form

$$\tfrac{1}{2}mv^2 + Ws = WH = \tfrac{1}{2}mv_0^2,$$

it asserts that the sum of the kinetic energy and the potential energy at any point is constant. This is the simplest example of the principle of the Conservation of Energy in its two mechanical forms of potential and kinetic energy. When the body leaves the ground the whole energy is in the kinetic form, and when it reaches the highest point it is all in the potential form.

## Motion on a Smooth Inclined Plane.

**302.** For a body moving on a smooth inclined plane, as in Fig. 84, the only force acting in the direction of the motion is the resolved part of the weight $W$ which acts down the plane. Denoting the inclination by $\theta$, this is $W \sin \theta$, hence the acceleration or force acting on a unit mass is $g \sin \theta$. Hence, for a body falling from rest, the equations, found as in Art. 295, are

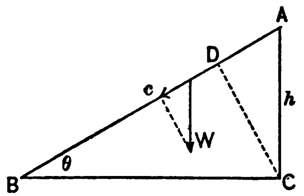


FIG. 84.

$$v = g \sin \theta . t, \quad \ldots \ldots \quad (1)$$

$$s = \tfrac{1}{2} g \sin \theta . t^2, \quad \ldots \ldots \quad (2)$$

$$v^2 = 2g \sin \theta . s, \quad \ldots \ldots \quad (3)$$

Let $AC = h$ be the height of the starting-point $A$ above the bottom of the plane, and $AB = c$ the length of the plane; then $h = c \sin \theta$. Putting $s = c$ in equation (3), we have then $v^2 = 2gh$; hence, comparing with Art. 295, we see that the velocity acquired by falling through the length of the plane is equal to that acquired by a body falling freely through the same height. Multiplying by $\frac{1}{2}m$, we have

$$\tfrac{1}{2}mv^2 = Wh,$$

or the kinetic energy acquired is, as before, equal to the work done by gravity.

## Spaces fallen through in Equal Times.

**303.** To compare the spaces described in the same time by the freely falling body and that on the inclined plane, let $t$ be the time occupied by the freely falling body in describing the space

$h$. By equation (2), Art. 295, $h = \frac{1}{2}gt^2$. Substituting in equation (2) above, we have

$$s = h \sin \theta.$$

This value of $s$ is $AD$ in Fig. 84, constructed by drawing $CD$ perpendicular to $AB$. Thus, if the bodies start from rest at the same instant, they will reach $C$ and $D$ respectively in the same time.

Suppose now that while $A$ and $C$ are fixed points, the inclination $\theta$ of the plane is varied. Because $ADC$ is a right angle, the locus of $D$ is a circle described on $AC$ as a diameter. Hence *the time of falling through any smooth chord drawn from the highest point of a vertical circle is the same as the time of falling through the vertical diameter.*

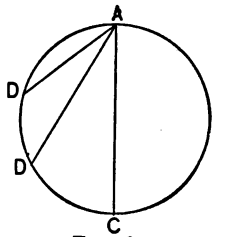

FIG. 85.

The same thing is obviously true of chords drawn to the lowest point of the circle.

The proposition may be used in the graphical solution of certain problems involving the straight line of quickest descent. For example, to construct the straight line of quickest descent from a given point $A$ to a given curve we have only to draw the smallest circle of which $A$ is the highest point and which meets the given curve. This circle is evidently tangent to the given curve.

## Body Projected up an Inclined Plane.

304. For a body projected up a smooth inclined plane, the initial circumstances being taken as in Art. 298, and the space measured up the plane, the equations become

$$v = v_0 - g \sin \theta \,.\, t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = v_0 t - \tfrac{1}{2} g \sin \theta \,.\, t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = v_0^2 - 2g \sin \theta \,.\, s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Denoting by $H$ the greatest vertical height to which the body will ascend, we have, putting $v = 0$ in equation (3),

$$H = \frac{v_0}{2g},$$

the same result as in Art. 300. Thus the body will rise to the same height as if it were projected vertically upward.

305. Multiplying equation (3) by $\frac{1}{2}m$, we have

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 - W \sin\theta \,.\, s,$$

or putting $s \,.\, \sin\theta = h$ (so that $h$ is the vertical height corresponding to the velocity $v$), and introducing $H$,

$$\tfrac{1}{2}mv^2 = W(H - h).$$

Hence, although the velocity is in a direction oblique to the force, the kinetic energy at any vertical height is the equivalent of the work done by gravity in the vertical distance of the body below the highest point.

Again, if the level of the point of projection be taken as that of zero-potential, we have

$$\tfrac{1}{2}mv^2 + Wh = WH = \tfrac{1}{2}mv_0^2,$$

which expresses that the sum of the potential and kinetic energies is constant. There is a continual transferrence of energy from the kinetic to the potential form and *vice versa*, but no loss of total energy.

## Motion on a Rough Plane.

306. Let us next suppose the plane to be rough, then when the body is moving down the plane with the same initial circumstances as in Art. 302, the friction acts up the plane, and its value is $-\mu R$, where $\mu$ is the coefficient of dynamical friction and $R = W\cos\theta$. See Fig. 84, p. 236. Therefore the acceleration

down the plane is $f = g(\sin\theta - \mu\cos\theta)$. The equations now become

$$v = g(\sin\theta - \mu\cos\theta)t, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$s = \tfrac{1}{2}g(\sin\theta - \mu\cos\theta)t^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = 2g(\sin\theta - \mu\cos\theta)s. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

These equations, of course, presuppose that the expression for $f$ is positive, so that motion actually takes place; that is, $\tan\theta > \mu$, or $\theta > \alpha$, the angle of friction. This being the case, suppose the body to fall from $A$, Fig. 84, to $B$; then, putting $s = c$, equation (3) gives, for the kinetic energy at the bottom of the plane,

$$\tfrac{1}{2}mv^2 = Wc\sin\theta - \mu Wc\cos\theta = Wh - \mu Wb, \quad . \quad (4)$$

where $b$ is the base $BC$ of the plane. Therefore the kinetic energy acquired in the fall is less than $Wh$, the potential energy expended, by the amount $\mu Wb$; this is therefore the energy expended in overcoming the non-conservative force of friction.

307. If the body is projected up the plane, the initial circumstances being as in Art. 304, the friction as well as the resolved part of the weight will act down the plane and

$$f = -g(\sin\theta + \mu\cos\theta).$$

The relations now become

$$v = v_0 - g(\sin\theta + \mu\cos\theta)t, \quad . \quad . \quad . \quad . \quad (1)$$

$$s = v_0t - \tfrac{1}{2}g(\sin\theta + \mu\cos\theta)t^2, \quad . \quad . \quad . \quad . \quad (2)$$

$$v^2 = v_0^2 - 2g(\sin\theta + \mu\cos\theta)s. \quad . \quad . \quad . \quad . \quad (3)$$

If $B$, Fig. 84, is the point of projection and $A$ the highest point reached, we find, by putting $v = 0$ in equation (3),

$$v_0^2 = 2gc(\sin\theta + \mu\cos\theta),$$

and multiplying by $\tfrac{1}{2}m$,

$$\tfrac{1}{2}mv_0^2 = Wh + \mu Wb.$$

Taking the potential energy as zero at the bottom of the plane, the first member expresses the total energy at the instant of projection. At the highest point $A$, this energy has been expended; the part $Wh$ has been converted into potential energy, and the remaining part, $\mu Wb$, has been used in doing work against friction.

**308.** The equations above apply only up to the time when the body reaches its highest point, because if the body descends friction will act up the plane, thus changing its direction. If $\theta > \alpha$, there will be a downward motion in accordance with the equations of Art. 306; but, if $\theta < \alpha$, the motion will cease. In particular, if $\theta = 0$, we have the case of a body projected along a rough horizontal plane. Such a body is subject to a retardation $\mu g$, and the space $s$, which will be described before the body comes to rest, is given by the equation

$$\tfrac{1}{2} m v_0^2 = \mu W s,$$

which expresses that the initial energy is all expended in work against friction.

EXAMPLES. XV.

1. If a body start with a velocity of 4 feet per second and move with one foot-second unit of acceleration, in what time will it acquire a velocity of 30 miles per hour? 40 seconds.

2. A stone skimming on ice passes a certain point with a velocity of 20 feet per second and suffers a retardation of one unit. Find the space described in the next 10 seconds, and the whole space described when the stone has come to rest.
150 ft.; 200 ft.

3. A body whose velocity is uniformly accelerated has at a certain instant a velocity of $22^f/_s$. In the following minute it travels 10,320 feet. Find the acceleration. $5^f/_{s^2}$.

4. A uniformly accelerated body passes two points 30 feet apart with velocities of 7 and 13 feet respectively. What is the acceleration? 2 ft.-sec. units.

5. A body whose motion is uniformly retarded changes its velocity from $24^f/_s$ to $6^f/_s$ while describing 12 feet. In what time does it describe the 12 feet? $\tfrac{4}{5}$ sec.

6. A steamer approaching a dock with engines reversed so as to produce a uniform retardation is observed to make 500 feet during the first 30 seconds of the retarded motion and 200 feet during the next 30 seconds. In how many more seconds will the headway be completely stopped? 5.

*In the following examples take $g = 32$ when numerical results are required:*

7. A body is let fall from a point 576 feet above the ground. With what velocity should another body be projected vertically upward from the same point and at the same instant, in order that it may strike the ground 4 seconds after the first body? $102.4^f/_s$.

8. A body is dropped from a height $AB = h$, and at the same moment a body is projected vertically upward from $B$. What must be the initial velocity if they are to meet half way?

$v_0 = \sqrt{(gh)}$.

9. To what height will a body projected upward with a velocity of 40 feet per second rise; and at the end of what times will it be 9 feet from the ground? 25 ft.; $\frac{1}{4}$ and $2\frac{1}{4}$ sec.

10. Two bodies are let fall from the same point at an interval of one second. How many feet apart will they be at the end of four more seconds? $\frac{9}{2}g$.

11. A body projected vertically upward remained for 4 seconds above the 960-foot level. What was the velocity of projection?

256 ft. per sec.

12. A balloon ascends with the uniform acceleration $\frac{1}{8}g$. At the end of half a minute a stone is dropped from it; how long will it take to reach the ground? 15 sec.

13. A ball is projected vertically upward with a velocity of 128 feet per second: when it has reached $\frac{3}{4}$ of its greatest height, another is projected from the same point with the same velocity. At what height will they meet? 240 feet.

14. A stone is dropped into a well, and the sound of the splash is heard 7.7 seconds afterward. Find the depth of the well, supposing the velocity of sound to be 1120 feet per second.

784 feet.

15. With what velocity in feet per second must a body be projected upward to reach the top of a tower 210 feet high in 3 seconds; and with what velocity will it reach the top?

118; 22.

16. A body projected upward from the top of a tower $a$ feet high reaches the ground 4 seconds later than a body dropped at the same time. What was its initial velocity?

$$v_0 = 4g\frac{\sqrt{a} + \sqrt{(2g)}}{\sqrt{a} + 2\sqrt{(2g)}}.$$

17. Show that the distance between two falling bodies in the same vertical line is a uniformly varying quantity. Thence find the velocity with which a body must be projected downward, to overtake in $t$ seconds a body which has fallen from rest at the same point through $a$ feet.

$$v_0 = \frac{a}{t} + \sqrt{(2ag)}.$$

18. A body is projected vertically downward from the top of a tower with the velocity $V$. One second afterwards another body is dropped from a window $a$ feet below the top. Determine in how many more seconds it will be overtaken by the first body, and explain the result when it becomes negative.

$$\frac{2a - 2V - g}{2(V+g)}.$$

19. A body is projected down a smooth inclined plane whose height is $\frac{1}{120}$ of its length with a velocity of $7\frac{1}{2}$ miles per hour Find the space passed over in two minutes. 3240 feet.

20. Show that the times of falling down smooth planes of the same height are proportional to the lengths of the planes.

21. A body weighing 30 pounds falls down a rough inclined plane of height 30 feet and base 100 feet. If $\mu = \frac{1}{5}$, what is the kinetic energy acquired? 300 foot-pounds.

22. A weight of 40 pounds is projected along a rough horizontal plane with a velocity of 150 feet per second. The coefficient of dynamical friction being $\frac{1}{8}$, what is the work done against friction in the first five seconds, and in the five seconds immediately preceding rest? 3500; 250 foot-pounds.

23. A train weighing 60 tons has a velocity of 40 miles an hour when the steam is shut off. If the resistance to motion is

10 pounds per ton, and no brakes are applied, how far will it travel before the velocity is reduced to 10 miles an hour?

11,293⅓ feet.

24. Show that the straight line of quickest descent from a point to a curve in the same vertical plane makes equal angles with the vertical and the normal at its extremity; and that the line of quickest descent between two curves makes the same angles with the two normals at its extremities.

25. Show how to construct graphically the straight line of quickest descent from a given point to a given circle.

26. What is the angular distance between the highest point of a vertical circle and the point from which the time down the radius is the same as the time down the chord to the lowest point?

$60^\circ$.

27. Show that, if the plane is rough, the locus of the point corresponding to $D$, Fig. 84, p. 236, is the arc of a circle, and that the locus of the point corresponding to $B$ (where the velocity of a body starting from rest at $A$ is the same as that of the freely falling body at $C$) is a straight line.

28. A heavy body projected up a rough plane whose inclination is $15^\circ$ came to rest in 5 seconds after sliding 200 feet along the plane. Find the coefficient of friction. $\mu = .2497$.

29. A body with constant acceleration acquires a velocity of $45^{m}/_{h}$ in ½ mile from rest. In what time is the ½ mile described?

80 sec.

## XVI.

### Kinetic Equilibrium.

**309.** We have seen that the inertia of a body undergoing acceleration may be regarded as a force balancing that which produces the acceleration. So also, when more than one force beside the inertia acts, we have, by including the inertia-force, a system of forces in equilibrium. In employing this principle,

which may be called that of *kinetic equilibrium*,* the unknown quantity derived from the condition of equilibrium may be a force instead of an acceleration. For example, suppose a man whose weight is $W$ to be standing on the floor of an elevator which begins to descend with the known acceleration $\alpha$. The forces acting on the man are his weight, $W = mg$, acting downward, his inertia, $m\alpha$, acting upward because the acceleration is downward, and the resistance $R$ of the floor of the elevator acting upward. Since the forces are all vertical, there is but one condition of equilibrium, namely, $W = R + m\alpha$.

Substituting $\frac{W}{g}$ for $m$, we find

$$R = W\left[1 - \frac{\alpha}{g}\right].$$

For example, taking $g = 32$, if $\alpha = 8$, we find $R = \frac{3}{4}W$; in other words, three-fourths of the man's weight is sustained by the floor, the other one-fourth going to produce the acceleration without which the man would not follow the elevator in its motion.

When the elevator has assumed a uniform velocity, $\alpha$ vanishes and $R = W$, exactly as if there were no motion. When the elevator is coming to rest, $\alpha$ changes sign in the equation as written above, because the acceleration has changed its direction. Hence, during the retardation, the pressure upon the floor is greater than the weight.

**310.** As a further illustration, suppose a brick of mass $m$ to be dragged over a rough horizontal table by means of a string parallel to the table. If the velocity is constant, the force exerted, which is the tension of the string, is equal to the dynamical friction. But, if the brick is to receive the acceleration $f$,

* This principle in its application to the general equations of motion is known as D'Alembert's Principle.

the tension must be increased by the amount $mf$ to overcome the resistance to acceleration, that is, the inertia. Again, suppose the tension to fall below the friction, the brick will be retarded, and until it comes to rest the inertia will act in the direction of motion and assist the tension in overcoming friction.

## Acceleration of Interacting Bodies.

**311.** When the mutual action of two bodies is such as to furnish a relation between their motions, the kinetic equilibrium of the two bodies may be used to determine at once their accelerations and their mutual action.

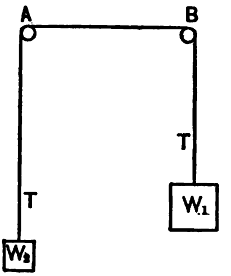

FIG. 86.

For example, if the weights $W_1$ and $W_2$ are connected by an inextensible string passing over two smooth pegs or pulleys, as in Fig. 86, the downward acceleration of the greater weight is evidently equal to the upward acceleration of the less. Their mutual action is the tension $T$ of the string, which acts upward in each case. Then, denoting the common acceleration by $\alpha$, we have to determine the two unknown quantities, $T$ and $\alpha$, by means of two equations, one derived from each of the bodies and expressing the kinetic equilibrium of vertical forces. When a single acceleration is involved, it is convenient to place upon one side of the equation the algebraic sum of all the external forces, regarding the direction of the acceleration as positive. The second member will then be the product of the mass and acceleration, which is in fact the inertia force acting in the opposite direction. Thus, in the present case, if $W_1 > W_2$, the acceleration of $W_1$ is downward; hence we write

$$W_1 - T = m_1\alpha = \frac{W_1}{g}\alpha, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

equating the excess of downward force to the inertia it overcomes. In like manner for the other body we have

$$T - W_2 = \frac{W_2}{g}\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating $T$, we derive

$$W_1 - W_2 = \frac{W_1 + W_2}{g}\alpha;$$

whence

$$\alpha = \frac{W_1 - W_2}{W_1 + W_2}g. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, eliminating $\alpha$ from equations (1) and (2),

$$W_2(W_1 - T) = W_1(T - W_2);$$

whence

$$T = \frac{2W_1W_2}{W_1 + W_2}.$$

This value of $T$ is intermediate in value between $W_1$ and $W_2$. It is in fact the so-called "harmonic mean" of these quantities.

**312.** Since the bodies $W_1$ and $W_2$ have the same speed, they may in a sense be regarded as a single mass which has the acceleration $\alpha$. The force producing this acceleration is then $W_2 - W_1$, and equating this to the product of the total mass into the acceleration we have

$$W_2 - W_1 = \frac{W_2 + W_1}{g}\alpha,$$

giving the equation (3) at once.

The arrangement shown in Fig. 86 is the essential part of Attwood's machine, by which the acceleration of gravity may be diminished in any chosen ratio, so that the velocity produced can be conveniently measured. Thus, if the two weights were each $15\frac{1}{2}$ ounces and an extra weight of one ounce were added to one

of them, we should have a total mass of two pounds moved by a force of one ounce, hence by equation (3) the acceleration will be $\frac{1}{32}$ of $g$.

## Application of the Principle of Work.

**313.** The principle of virtual work, or of work-rate, is sometimes employed when one of the forces in question is that of inertia. For example, a train weighing 160 tons is hauled up a grade of 1 in 140, the resistances from friction, etc., being 12 pounds per ton. Required to find the acceleration at the instant the speed is 15 miles an hour, if the engine is then developing 200 horse-power, that is to say, doing work at the rate of $200 \times 550$ foot-pounds per second.

This work is done against the resistance $R$, the component of the weight along the inclined plane, which is $\frac{1}{140} W$, and the inertia $m\alpha$. Since the speed is $22\,{}^{f}/_{s}$, the space through which the sum of these forces works in one second (or rather the rate per second at which they are at the instant working) is 22. Hence

$$200 \times 550 = 22\left(12 \times 160 + \frac{160 \times 2240}{140} + \frac{160 \times 2240}{g}\,\alpha\right),$$

whence we find $\alpha = \frac{13}{280}$.

It is obvious that the process is equivalent to equating the forces which act at the two ends of the draw-bar, since the force is the result of dividing the work by the space through which the force acts.

**314.** In Section XIV. we have employed the total work done in a displacement in solving questions involving forces and spaces only. Such questions usually imply the transferrence of a mass from one position to another, which generally brings into action the force of inertia. Thus, if the initial position is one of rest, some motion, and therefore some acceleration, must take place. We have seen in Art. 294 that the work done against inertia takes the form of kinetic energy, and that during retarda-

tion an amount of work is done by inertia equivalent to the loss of kinetic energy. Hence, if the final position is also one of rest, the force of inertia does not appear in the equation derived from the total work.

315. But, in applying the principle to the more general case where the initial and final circumstances involve velocities, the changes in kinetic energy must be reckoned as part of the work done. As an illustration, take the following example: A body $A$, weighing 1 pound, is connected with a body $B$, weighing 2 pounds, by means of a string which passes over a smooth pulley at the edge of a rough horizontal table on which $B$ rests, while $A$ hangs at a distance of 18 inches from the floor. Supposing $\mu = \frac{1}{4}$, if $A$ is allowed to drop, find the distance $s$ which $B$ will travel after $A$ strikes the floor.

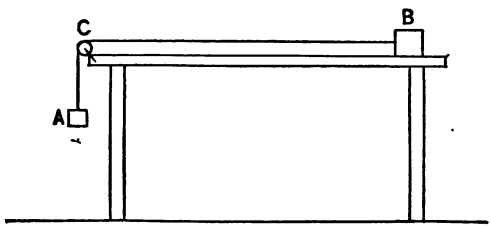


FIG. 87.

The whole work done by gravity upon the system consisting of the two weights is here $1\frac{1}{2}$ foot-pounds. The friction $F$ is $\mu$ times the weight of $B$, that is, $\frac{1}{2}$ pound. When $A$ reaches the floor, $B$ has moved $1\frac{1}{2}$ feet, and the work done against friction is $\frac{3}{4}$ of a foot-pound. There remains $\frac{3}{4}$ of a foot-pound at this instant in the form of kinetic energy; and since $A$ has one-half the mass of $B$, one-third of this kinetic energy or $\frac{1}{4}$ foot-pound is in $A$, and two-thirds or $\frac{1}{2}$ foot-pound in $B$. The former is lost (that is to say, disappears from its mechanical forms), while the latter is exhausted in doing the work $Fs$ against friction. Therefore $Fs = \frac{1}{2}$, whence $s = 1$.

## Resolution of Inertia Forces.

316. When bodies are subject to accelerations in different directions, the corresponding inertia forces may be treated ex-

actly like other forces in deriving equations of equilibrium by the resolution of forces, as in the following example:

A smooth isosceles wedge of mass $M$, Fig. 88, and base angle $\alpha$ rests on a smooth horizontal plane and carries on its two inclined faces bodies of masses $m_1$ and $m_2$ (of which $m_1 > m_2$), which are connected by a string which passes over a smooth pulley at the top of the wedge; it is required to find the

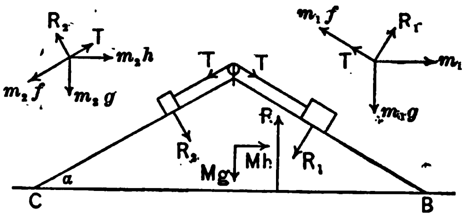


FIG. 88.

acceleration of the wedge, and the acceleration of the masses relatively to the wedge. Let $f$ denote this last acceleration, that is to say, the rate of change of the speed with which the string passes over the pulley. Let $h$ denote the horizontal acceleration of the wedge toward the left, which is also shared by the masses $m_1$ and $m_2$; and let $T$, $R$, $R_1$ and $R_2$ be the tension of the string, the resistance of the horizontal plane, and the actions between the wedge and the masses $m_1$ and $m_2$.

To determine these six unknown quantities we have two conditions of equilibrium for each of the bodies $m_1$, $m_2$ and $M$. The forces acting on $m_1$ and $m_2$ respectively are shown in separate diagrams for clearness. Taking, in each of these cases, forces along, and perpendicular to, the face of the wedge, we have

$$R_1 = m_1 g \cos \alpha - m_1 h \sin \alpha, \quad \ldots \ldots \quad (1)$$

$$T = m_1 g \sin \alpha + m_1(h \cos \alpha - f), \quad \ldots \quad (2)$$

$$R_2 = m_2 g \cos \alpha + m_2 h \sin \alpha, \quad \ldots \ldots \quad (3)$$

$$T = m_2 g \sin \alpha - m_2(h \cos \alpha - f). \quad \ldots \quad (4)$$

The forces acting upon $M$ are its weight, its inertia $Mh$ acting as before to the right, the reactions of the resistances and

tensions which act on $m_1$ and $m_2$, and the resultant upward resistance $R$ of the fixed plane. Hence, taking horizontal and vertical forces acting on $M$,

$$Mh = (R_1 - R_2) \sin\alpha, \quad \ldots \ldots \ldots \ldots \quad (5)$$

$$R = Mg + (R_1 + R_2)\cos\alpha + 2T\sin\alpha. \quad \ldots \quad (6)$$

317. Eliminating $R_1$ and $R_2$ by means of equations (1) and (3), equation (5) gives

$$h = \frac{(m_1 - m_2)g \sin\alpha \cos\alpha}{M + (m_1 + m_2)\sin^2\alpha};$$

and the substitution of this value gives

$$R_1 = m_1 g \cos\alpha \frac{M + 2m_2 \sin^2\alpha}{M + (m_1 + m_2)\sin^2\alpha},$$

$$R_2 = m_2 g \cos\alpha \frac{M + 2m_1 \sin^2\alpha}{M + (m_1 + m_2)\sin^2\alpha}.$$

Again, eliminating $h$ and $f$ from equations (2) and (4), we find

$$T = \frac{2m_1 m_2 g \sin\alpha}{m_1 + m_2};$$

and, this being substituted in equation (2), we obtain

$$f = h\cos\alpha + g\sin\alpha - \frac{2m_2 g \sin\alpha}{m_1 + m_2},$$

which, when the value of $h$ found above is substituted, becomes

$$f = \frac{(m_1 - m_2)(M + m_1 + m_2)g \sin\alpha}{(m_1 + m_2)[M + (m_1 + m_2)\sin^2\alpha]}.$$

Finally, the value of $R$ is most conveniently found by first substituting in equation (6) the values of $R_1$ and $R_2$ directly

from equations (1) and (3), and that of $2T$ which results from adding equations (2) and (4); thus

$$R = (M + m_1 + m_2)g - (m_1 - m_2)f \sin \alpha,$$

in which $f$ has the value given above. This equation shows that the pressure on the plane is less than it would be if motion were prevented, the diminution being the excess of the upward component of the inertia of $m_1$ over the downward component of the inertia of $m_2$.

EXAMPLES. XVI.

1. If the weight of a balloon and its appendages is 4500 pounds, and that of the air displaced (which is the upward force) is 4800 pounds, with what acceleration does it begin to ascend? $\frac{1}{16}g$.

2. Two bodies weighing 3 pounds each are connected by a light string passing over a smooth peg. If a third body of the same weight is added to one of them, how much is the pressure on the peg increased? 2 pounds.

3. Two weights of 5 and 4 pounds respectively are attached to one end of a string which passes over a smooth pulley and has a weight of 7 pounds on the other end. The two weights descend through a distance $s$, and the 4-pound weight is then detached. How much farther will the 5-pound weight descend. $\frac{3}{4}s$.

4. In an Attwood's machine a 40-gramme weight is drawn up by a 50-gramme weight 2.18 metres in 2 seconds. What is the value of $g$ in centimetres per second per second? 981.

5. If a 3-pound weight hanging over the edge of a smooth horizontal table drags a 45-pound weight along it, determine the acceleration and the tension of the string.

$\alpha = \frac{1}{16}g$; $T = 2$ lbs. 13 oz.

6. Two equal weights are connected by a string 7 feet in length, one of them resting upon a smooth horizontal table, 3 feet high, at a point 6 feet from the edge, where the string passes over a

smooth pulley to the other weight hanging freely. In what time from rest will the first weight reach the edge of the table?

1 second.

7. If, in the preceding example, the table is so rough that the body just reaches the edge, in what time will it do so?

$\frac{3}{4}\sqrt{5}$ sec.

8. If the string in Fig. 86, p. 245, can only sustain a tension of $\frac{1}{4}$ of the sum of the weights, show that the least possible value of the acceleration is $\frac{1}{2}g\sqrt{2}$.

9. A train weighing 100 tons is drawn on a level track by a locomotive developing 150 horse-power, and the resistance is 14 pounds per ton. What is the acceleration when the train is moving 15 miles an hour?

$\frac{47g}{4480}$.

10. A bicyclist and his machine weigh 180 pounds. What horse-power does he exert in riding on a level track, whose resistance is one per cent of the weight, at the rate of 20 miles per hour?

.0096.

11. Weights of 11 and 5 pounds are suspended from the extremities of a cord which passes over a smooth fixed pulley. What is the velocity of either weight at the end of 5 seconds from rest, and the pressure on the supports of the pulley?

$60^f/_s$; $13\frac{3}{4}$ lbs.

12. Two unequal weights, $W_1 > W_2$, on a rough inclined plane are connected by a string passing through a smooth pulley fixed to the plane so that the parts of the string are parallel to the plane. Determine the acceleration $f$, the inclination being $\theta$, and the coefficient of friction $\mu$.

$$f = g\left[\frac{W_1 - W_2}{W_1 + W_2}\sin\theta - \mu\cos\theta\right].$$

13. A mass $A$ draws a mass $B$ up a smooth inclined plane by means of a string passing over the vertex. Determine the inclination of the plane so that $A$ may draw $B$ up a given vertical height in the shortest possible time.

$$\theta = \sin^{-1}\frac{A}{2B}.$$

14. The height of an inclined plane is 5 feet and its length 13

feet. A weight of 10 pounds is suspended from one end of a cord which passes over a smooth pulley at the top of the plane and is attached to a weight of 3 pounds resting on the plane. Find the tension of the string during motion: 1st, if the plane is smooth; 2d, if $\mu = \frac{1}{3}$. 3.20 lbs.; 3.90 lbs.

15. A string passing over a fixed pulley carries a weight of 2 pounds on one end and a pulley on the other, over which passes a string carrying a weight of one pound at each end. The system being at rest in equilibrium, a force is applied to one of the one-pound weights. Prove that when it has moved the weight down three inches each of the other weights has risen one inch.

16. A train weighing 100 tons is ascending a 1-per-cent grade, and the frictional resistance is 12 lbs. per ton. What is its greatest speed, if the engine can develop 200 horse-power at that speed? 21.8 $^{m}/_{h}$.

17. A weight of 10 pounds rests on a rough horizontal table, $\mu = \frac{1}{5}$; a string attached to it passes over a smooth pulley at the edge of the table to an equal weight at the distance of $2\frac{1}{2}$ feet from the floor. If this second weight is let fall, find the tension of the string, the time to reach the floor, and how much farther the weight on the table will go. 6 lbs.; $\frac{5}{8}$ sec.; 5 feet.

18. Two weights $P$ and $Q$ are connected by a string which passes over a pulley at the top of a smooth plane inclined 30° to the horizon. $Q$, hanging freely, can draw $P$ up the length of the plane in half the time that $P$ would take to draw $Q$ up. Find the ratio of $Q$ to $P$. 3 : 2.

## XVII.

### Motion Oblique to the Direction of the Force.

**318.** In this section, we suppose the particle, acted upon by a force constant in direction as well as magnitude, to have an initial motion oblique to the direction of the force. The most important application is to the case in which the force is the weight of a particle, or the weight of a body regarded as acting at the centre of inertia. We therefore take as the representative case the motion of a *projectile* or body projected from a point in a direction not vertical. The plane of projection is the vertical plane through the line of projection, which is a tangent to the path of the projectile. Since there is no component of force tending to move the body out of this plane, the path, which is called *the trajectory*, will be a plane curve lying in this plane.

**319.** Let us first suppose the body to be moving through the point $O$, Fig. 89, with the velocity $V$ in the direction of the horizontal line $Ox$. Then, if gravity did not act, the particle would at the end of one second arrive at the point $A_1$, where $OA_1 = V$; at the end of two seconds at $A_2$, where $OA_2 = 2V$; and at the end of any time $t$ at $A$, where $OA = Vt$. Resolving the actual velocity of the particle $P$ at any instant into its horizontal and vertical components, there is no force tending to disturb the horizontal velocity; hence, at the instants mentioned

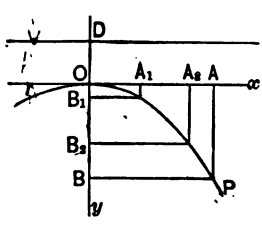


FIG. 89.

above, the particle $P$ will actually be found at points vertically below the points $A_1$, $A_2$ and $A$ respectively. Consider now the vertical velocity. This is zero at the point $O$, at which point, therefore, the trajectory is tangent to $Ox$. Now, since the vertical force is not affected by the horizontal motion, the vertical velocity is at every instant the same as that of a particle falling freely from $O$, in the vertical line $Oy$. It follows that, if $B_1$, $B_2$

and $B$ are the positions of the freely falling body at the instants 1, 2 and $t$, $OB_1$, $OB_2$ and $OB$ will be the actual distances of the particle at these instants from the horizontal line $Ox$; hence the actual positions of the particle will be as indicated in the diagram.

**320.** Referring the position of $P$ to $Ox$ and $Oy$ as axes of coordinates, we have at the time $t$

$$x = Vt, \qquad (1)$$

and, by Art. 295,

$$y = \tfrac{1}{2}gt^2. \qquad (2)$$

These equations connect the position of the body with the time, and suffice to solve such questions as the following:

A body is projected horizontally with a velocity of $20^f/_s$ from a tower standing 128 feet above a horizontal plane; when and where will it strike the ground? Putting $y = 128$ in equation (2), and taking $g = 32$, we find $t = 2\sqrt{2}$; whence equation (1) gives $x = 40\sqrt{2}$; that is, the body hits the ground about 56.57 feet from the foot of the tower.

To find the equation of the trajectory, or direct relation between $x$ and $y$, we have, by eliminating $t$ from equations (1) and (2),

$$y = \frac{g}{2V^2}x^2, \qquad (3)$$

which is the equation of a parabola with its axis vertical.

## Parabolic Motion.

**321.** Before proceeding to the equation of the trajectory referred to the point of projection, we shall use the symmetrical form of the equation found above in deriving some properties of the motion. Let $H$ denote the height due to the velocity $V$ (see Art. 297), then

$$V^2 = 2gH, \quad \text{and} \quad H = \frac{V^2}{2g}.$$

Substituting in equation (3), the equation of the curve takes the form

$$x^2 = 4Hy.$$

Comparing this with the usual form of the equation of the parabola, we see that $4H$ is the parameter or latus-rectum, and $H$ is the distance of the vertex from the focus or from the directrix. Hence, measuring $OD = H$ vertically upward in Fig. 89, the horizontal line through $D$ is the directrix.

**322.** The horizontal and vertical velocities of the particle are denoted by $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively, and, by Art. 43, the actual speed in the curve is given by

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the present case, $\frac{dx}{dt} = V$, and, by Art. 295, $\left(\frac{dy}{dt}\right)^2 = 2gy$.

Substituting,

$$v^2 = V^2 + 2gy = 2g(H + y). \quad . \quad . \quad . \quad . \quad (2)$$

Now, in Fig. 89, $H + y$ is the distance of the particle $P$ below the directrix ; hence, by Art. 297, *the velocity at any point is equal to that due to the distance of the point below the directrix.*

## Kinetic Energy of the Projectile.

**323.** If we multiply the general equation (1) of the preceding article by $\frac{1}{2}m$, we derive

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2$$

$v_x$ and $v_y$ denoting any pair of *rectangular* components of the

velocity. This equation shows that the actual kinetic energy of a body is the sum of the kinetic energies it would have if moving with one and then the other of its resolved velocities. The kinetic energy can thus, as it were, be resolved into two component parts, *but only when the component velocities are rectangular.*

Treating the particular equation (2) in like manner, we have

$$\tfrac{1}{2}mv^2 = W(H + y);$$

that is, the kinetic energy in the trajectory is the sum of the constant term $WH$, due to the constant horizontal velocity, and the variable part $Wy$, due to the vertical velocity.

## The Trajectory referred to the Point of Projection.

324. When a body is projected obliquely upward from a point on the ground taken as origin, it is convenient to measure $y$ upward. The equations of motion for the two component motions, which, as we have seen, may be considered separately, are

$$\frac{d^2x}{dt^2} = 0, \quad \text{and} \quad \frac{d^2y}{dt^2} = -g. \quad . \quad . \quad . \quad . \quad (1)$$

Let $O$, Fig. 90, be the point of projection, $V$ the velocity of projection, or initial velocity, and $\alpha$ the inclination of the line of projection to the horizontal. The initial horizontal and vertical velocities are then

$$V \cos \alpha \quad \text{and} \quad V \sin \alpha.$$

The first integration of equations (1) gives

$$\frac{dx}{dt} = V \cos \alpha, \quad \frac{dy}{dt} = V \sin \alpha - gt, \quad . \quad . \quad . \quad (2)$$

in which the constants of integration are the initial values which correspond to $t = 0$. A second integration gives, since $x = 0$ and $y = 0$ when $t = 0$,

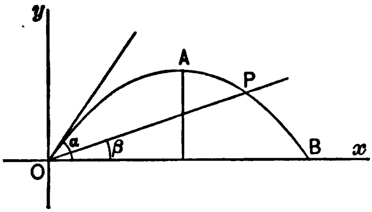


FIG. 90.

$$\left.\begin{aligned} x &= V\cos\alpha \,.\, t, \\ y &= V\sin\alpha \,.\, t - \tfrac{1}{2}gt^2. \end{aligned}\right\} \quad (3)$$

The value of $y$ shows that the vertical distance of the particle $P$ below the tangent at $O$ is that through which it would have fallen freely in the time of describing the arc $OP$.

**325.** Eliminating $t$ between equations (3), we have, for the equation of the trajectory

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we put, as in Art. 321,

$$V^2 = 2gH, \qquad H = \frac{V^2}{2g},$$

so that $H$ is the height due to the initial velocity, the equation may also be written

$$y = x\tan\alpha - \frac{x^2}{4H\cos^2\alpha}, \quad . \quad . \quad . \quad (5)$$

in which $H$ is now, by Art. 322, the distance of the point $O$ (not of the vertex) below the directrix; that is, the height of the directrix above the ground.

**326.** Equating to zero the vertical velocity, equation (2), we have

$$t = \frac{V\sin\alpha}{g}$$

for the time in which the initial vertical velocity is destroyed by gravity; and using this value of $t$ in equations (3), we find

$$x = \frac{V^2 \sin\alpha \cos a}{g}, \qquad y = \frac{V^2 \sin^2\alpha}{2g}$$

for the coordinates of the point $A$, Fig. 90, which is the highest point reached by the projectile, that is, the vertex of the parabola. Substituting the value of $V^2$ in the preceding article, these coordinates may be written

$$x = H \sin 2\alpha, \qquad y = H \sin^2\alpha.$$

327. The height corresponding to the initial vertical velocity $V \sin\alpha$ is $\dfrac{V^2 \sin^2\alpha}{2g}$, equal, as we should expect, to the height of the vertex. Again, the height corresponding to the constant horizontal velocity $V \cos\alpha$ is

$$H \cos^2\alpha.$$

This is, by Art. 322, the height of the directrix above the vertex. Accordingly the result of adding this to the height of the vertex is $H$, the height of the directrix above $O$. Since the vertex is equally distant from the focus and the directrix, the height of the focus is

$$H(\sin^2\alpha - \cos^2\alpha) = -H\cos 2\alpha.$$

The focus is therefore below, in or above the horizontal plane through the point of projection, according as $\alpha$ is less than, equal to or greater than 45°.

### The Range and the Time of Flight.

328. The distance from the point of projection at which the projectile strikes the horizontal plane ($OB$, Fig. 90) is called *the horizontal range*, and the time occupied in describing the arc $OAB$ is called *the time of flight*. The range and the time of flight are the values of $x$ and $t$ (distinct from zero) corresponding

to $y = c$ in equations (4) and (3); denoting them by $R$ and $T$ respectively, we have

$$R = \frac{V^2 \sin 2\alpha}{g} = 2H \sin 2\alpha, \qquad T = \frac{2V \sin \alpha}{g},$$

which are, as we should expect, double the values of $x$ and of $t$ at the vertex $A$.

The expression for $R$ shows that, for a given value of $V$, *the range has its maximum value*, $2H$, *when* $\alpha = 45°$; the focus is then, by the preceding article, in the horizontal plane. For values of $\alpha$ differing equally from $45°$, say $45° \pm \gamma$, we find the same value, namely, $R = 2H \cos 2\gamma$. This determines, for any desired value of the range (less than the maximum), two parabolas which will give that range.

329. *The range and the time of flight for an inclined plane* passing through the point of projection are determined by the intersection of the parabola with the straight line $OP$, Fig. 90, of which the equation is

$$y = x \tan \beta,$$

where $\beta$ is the inclination of the plane to the horizon. Substituting this value of $y$ in equation (4), Art. 325, we find (beside $x = 0$, corresponding to the point $O$)

$$\begin{aligned} x &= \frac{2V^2 \cos^2 \alpha}{g} (\tan \alpha - \tan \beta) \\ &= \frac{2V^2 \cos \alpha}{g \cos \beta} \sin (\alpha - \beta), \quad \dots \dots \quad (1) \end{aligned}$$

which is therefore the abscissa of $P$. From $x = Vt \cos \alpha$ we obtain, for the corresponding value of $t$,

$$T' = \frac{2V}{g \cos \beta} \sin (\alpha - \beta), \quad \dots \dots \quad (2)$$

the time of flight for the arc $OAP$.

Denoting the range $OP$ by $R'$, we have $R' = x \sec \beta$, whence

$$R' = \frac{4H}{\cos^2 \beta} \cos \alpha \sin (\alpha - \beta)$$

$$= \frac{2H}{\cos^2 \beta} [\sin(2\alpha - \beta) - \sin \beta]. \quad . \quad . \quad . \quad (3)$$

**330.** From the last expression it is evident that, for given values of $V$ and $\beta$, $R'$ is a maximum when $2\alpha - \beta = 90°$, or

$$\alpha = \tfrac{1}{2}(90° + \beta).$$

Hence the value of the maximum range is

$$\frac{2H}{1 + \sin \beta}.$$

The value of $\alpha$ shows that *for the maximum range the line of projection bisects the angle between the plane and the vertical.* Since, by a property of the parabola, this line also bisects the angle between the vertical and the line joining $O$ to the focus, it follows that, in this case, as well as in that of the maximum range on the horizontal plane, the focus lies in the plane.

### With a Given Initial Velocity to Hit a Given Point.

**331.** Let $P$, Fig. 90, be a given point, and denote the distance $OP$ by $R'$, and the angle $POB$ by $\beta$. If the trajectory is to pass through $P$, and the initial velocity is also given, we have, with the notation of the preceding articles, to determine the value of $\alpha$ for given values of $H$, $R'$ and $\beta$. Equation (3) of Art. 329 may be written

$$\sin (2\alpha - \beta) = \sin \beta + \frac{R' \cos^2 \beta}{2H}; \quad . \quad . \quad . \quad (1)$$

hence, if we put

$$\cos 2\gamma = \sin\beta + \frac{R'\cos^2\beta}{2H}, \quad . \quad . \quad . \quad (2)$$

we shall have
$$2\alpha - \beta = 90^\circ \pm 2\gamma,$$

or
$$\alpha = \tfrac{1}{2}(90^\circ + \beta) \pm \gamma. \quad . \quad . \quad . \quad . \quad (3)$$

Thus, in order that the trajectory should pass through the given point, $\alpha$ must have one of two values, each differing from that which gives the maximum range for the given value of $\beta$ by an angle $\gamma$ determined by equation (2).

When the given point is in the horizontal plane, $\beta = 0$, and the result reduces to that already found in Art. 328.

332 .The problem is impossible if the given distance $R'$ exceeds the maximum range (see Art. 330) for the given value of $\beta$, the value found for $\cos 2\gamma$ being in that case greater than unity. In other words, we cannot hit a point which lies beyond the locus of the point of maximum range for a variable value of $\beta$. To determine this locus, denote the maximum range by $r$; by Art. 330, we have

$$r = \frac{2H}{1 + \sin\beta},$$

or, since the ordinate of the extremity of $r$ is $y = r\sin\beta$ (see Fig. 90),

$$r = 2H - y.$$

Now $2H - y$ is the distance of the point below a horizontal line at the height $2H$ above $O$; hence the locus in question is a parabola, whose focus is the point of projection, and whose vertex is the point $(0, H)$ the highest point which can be reached with the velocity $V$. This parabola is therefore the boundary of the region of points which can be reached with the initial velocity $V$. In other words, it is *the envelope of the system of trajectories having the same initial velocity and point of projection.* Geometri-

cally considered, the enveloped curves are a system of parabolas passing through a common point and having a common directrix.

## Constant Value of the Total Energy.

**333.** We have seen in Art. 322 that the velocity of the particle at any point of its path is that due to its distance below the directrix. That is to say, in the notation of Art. 324, it is given by

$$v^2 = 2g(H - y),$$

a result which might have been obtained directly from equations (2) and (3) of that article.

Multiplying by $\frac{1}{2}m$, we have

$$\tfrac{1}{2}mv^2 + Wy = WH.$$

Taking the level of the point of projection as that of zero-potential, $Wy$ is the potential energy, and $\frac{1}{2}mv^2$ the kinetic energy. Hence the equation expresses the fact that these have a constant sum. The initial kinetic energy $\frac{1}{2}mV^2$, which is the total energy (all in the kinetic form at $O$), is the same as the potential energy of the body when at rest on the directrix.

### EXAMPLES. XVII.

1. A body projected horizontally from a height of 9 feet from the ground reached the ground at a horizontal distance of 120 feet. What was its initial velocity? 160 ft. per sec.

2. From the top of a cliff 80 feet high a stone is thrown with a velocity of 128 feet per second and an angle of elevation of 30°. Find at what distance from the bottom of the cliff it hits the ground. $320\sqrt{3}$ feet.

3. A body projected at an inclination of 45° to the horizon from the top of a tower fell in 5 seconds at a distance from the foot equal to the height of the tower. Find the height and the initial velocity. 200 feet.; $40\sqrt{2}$ ft. per sec.

4. If two equal bodies are projected with the same velocity at the two angles which give the same horizontal range, show that the sum of their kinetic energies at their highest points is independent of the angles of projection.

5. At what angles of projection is the horizontal range equal to the height due to the velocity? $15^\circ$ and $75^\circ$.

6. A piece of ice is detached on a roof whose slope is $30^\circ$ at a point 8 feet from the eaves, which are 24 feet above the ground. At what distance from the vertical plane through the eaves will it reach the ground? $8\sqrt{3}$ feet.

7. A projectile fired from the top of a tower at an angle of elevation of $45^\circ$ strikes the ground 60 feet from the foot of the tower at the end of 4 seconds. Find the height of the tower.

196 feet.

8. A ship is moving with the velocity $u$, and a ball is fired from a gun on deck with a charge which would give the velocity $v$ if the ship were at rest, at an elevation $\alpha$ in the vertical plane in which the ship is moving. Find the horizontal range.

$$\frac{2}{g} v \sin \alpha \, (v \cos \alpha \pm u).$$

9. Show that to hit a vertical wall squarely at a certain point we should aim at a point at double the height.

10. Show that the time of describing any arc of a trajectory is twice the time of a body falling from rest in the arc to the middle point of the chord.

11. A body is projected with the velocity $3g$ feet per second at an inclination of $75^\circ$ to the horizon. Find the range on a horizontal plane.

$$\frac{9}{2}g.$$

12. Show that the greatest range on a plane inclined $30^\circ$ to the horizon is two-thirds of the greatest horizontal range with the same velocity.

13. A body projected at an angle $\alpha$ just clears two walls of height $a$ and distance apart $2a$. Prove that the range is $2a \cot \frac{1}{2}\alpha$.

14. A stone thrown with a velocity of 64 feet per second is

to hit an object on top of a wall 19 feet high and 48 feet distant. Determine the value of $\alpha$.

$$\tan \alpha = \frac{2}{3} \text{ or } \frac{14}{3}.$$

15. A body is projected from the top of a tower whose height is $h$ with the velocity due to a height $nh$. Determine the greatest distance from the foot of the tower which can be reached, and the corresponding value of $\alpha$.

$$x = 2h\sqrt{(n^2 + n)}; \ \tan \alpha = \frac{\sqrt{n}}{\sqrt{(n + 1)}}.$$

16. If two bodies be projected from the same point with the same velocity and an inclination of 30°, one describing a free trajectory and the other moving up a smooth tangent to the point of rest, and thence falling freely, show that they will reach the horizontal plane of projection at the same point, the latter occupying an interval of time three times as great as the former does.

17. A particle is so projected as to enter a smooth straight tube of length $l$ in the direction of its length, which is inclined at the angle $\alpha$ to the horizon. Show that, if the particle passes through the tube, the height to which it will rise will exceed that to which it would have risen had there been no tube by the distance $l \sin \alpha \cos^2 \alpha$.

18. A projectile has an initial velocity of 280 feet per second. Find the angle of elevation in order that it may hit an object whose altitude is 192 feet and horizontal distance 2240 feet.

$$\alpha = 45° \text{ or } 49° \ 54'.$$

19. Denoting by $T'$, as in Art. 329, the time of flight for an inclined plane, show that the space through which the body falls from the initial tangent is

$$\frac{VT' \sin (\alpha - \beta)}{\cos \beta},$$

and thence derive the values of $T'$ and the range $R'$.

20. When the range is a maximum, show that the time of flight is

$$T' = \frac{V}{g \cos \delta},$$

where $\delta$ is the inclination of the line of projection to the plane.

21. Show from the result of the preceding example that the time of describing any arc whose chord passes through the focus is

$$\frac{2V_0}{g \sin 2\alpha},$$

where $V_0$ is the horizontal velocity, and $\alpha$ is the inclination at either end of the arc.

22. Show that in any trajectory the velocities at the two extremities of a focal chord are as the sines of the inclinations to the horizon, and also as the intercepts of the tangents between the points of contact and the directrix.

23. If particles be projected with the same velocity in different directions in the same vertical plane, prove that the time in which any particle reaches the envelope of the trajectories is the same as that in which it would have come to rest if projected up a smooth tangent to its trajectory.

24. All particles projected as in the last example would, if gravity did not act, at the end of the time $t$ lie upon the circumference of a circle whose radius is $Vt$ and centre at the origin; therefore they are actually upon a circle with the same radius and centre at the point $(0, -\frac{1}{2}gt^2)$. Show that the envelope of the trajectories is also the envelope of this circle when $t$ varies; and thence find its equation. $x^2 = -4H(y - H)$.

25. Show that the locus of the vertices of the trajectories in Ex. 23 is an ellipse whose semi-axes are $H$ and $\frac{1}{2}H$.

26. Show that the boundary of the area on a plane inclined at the angle $\beta$ which can be reached by a projectile with given velocity from a given point of projection in the plane is an ellipse whose eccentricity is $\sin \beta$.

27. Prove that the straight line joining the positions at any instant of two bodies projected at the same time from the same point and with the same initial velocity is perpendicular to the line bisecting their initial directions.

28. Show that the hodograph of parabolic motion is a vertical straight line described with uniform motion.

# CHAPTER IX.

## MOTION PRODUCED BY A VARIABLE FORCE.

### XVIII.

### Rectilinear Motion.

334. When the force acting upon a body is a function of its distance $s$ from a fixed centre of force, and there is no motion transverse to the line of action, the acceleration or force acting on a unit mass may be denoted by $f(s)$, where $f$ stands for a given function. The single equation of motion now takes the form

$$\frac{d^2s}{dt^2} = f(s).$$

We have seen in Art. 291 that this is equivalent to the two equations,

$$dv = f(s)dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$ds = vdt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

between the three variables $s$, $v$ and $t$, and that by the elimination of $dt$ we have also the third relation,

$$vdv = f(s)ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (1), containing three variables, is not now directly integrable, as it was when $f$ was constant; but, as mentioned in Art. 294, equation (3) is still directly integrable because it contains but two variables. Hence *in all cases of variable force this*

*is the equation first integrated.* The result gives $v$ in terms of $s$, and this substituted in equation (2) gives a differential relation between $s$ and $t$ for the second integration.

## Attractive Force Varying Directly as the Distance.

**335.** As a first example, let us take the case of a particle attracted to a fixed point by a force directly proportional to the distance. Taking the fixed point or centre of force $O$, Fig. 91, for origin, let the body start from rest at the point $A$ on the positive side at the distance $OA = a$. Then, taking the corresponding instant as the origin of time, the initial circumstances of the motion are $t = 0$, $v = 0$, $s = a$. Since the force is attractive, the acceleration is negative when $s$ is positive; and, since it is proportional to $s$, its value is $-\mu s$, where the positive quantity $\mu$ is the intensity of the force at a unit's distance. Equation (3) of Art. 334 thus becomes

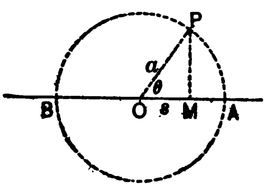


FIG. 91.

$$vdv = -\mu s\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating, we have

$$v^2 = -\mu s^2 + v_0^2,$$

where the constant of integration $v_0$ is the velocity at the origin. It will be convenient to put $n^2$ for the positive quantity $\mu$. Then, since $v = 0$ when $s = a$,

$$v_0^2 = n^2 a^2,$$

and we have for the value of $v$ in terms of $s$

$$v^2 = n^2(a^2 - s^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

336. Substituting $\frac{ds}{dt}$ for $v$ in equation (2), and separating the variables, we have

$$\frac{ds}{\sqrt{(a^2 - s^2)}} = n dt. \quad \ldots \ldots \ldots \quad (3)$$

Hence the second integration gives

$$\sin^{-1}\frac{s}{a} = nt + C.$$

By the initial circumstances, $t = 0$ must give $s = a$; hence $C$ is a value of the multiple-valued function $\sin^{-1}1$. Taking $\frac{1}{2}\pi$ for this value, the equation becomes

$$\frac{s}{a} = \sin(nt + \tfrac{1}{2}\pi) = \cos nt,$$

or

$$s = a \cos nt. \quad \ldots \ldots \ldots \quad (4)$$

Finally, differentiation of this value of $s$ with respect to $t$ gives

$$v = -na \sin nt. \quad \ldots \ldots \ldots \quad (5)$$

Equations (2), (4) and (5) give the relations between each pair of the variables $s$, $v$ and $t$.

337. Let $MP$, Fig. 91, be a perpendicular to $AB$, erected from the position $M$ of the particle at the time $t$, and meeting the circle whose centre is $O$ and radius $a$ in $P$, join $OP$; then, denoting the angle $AOP$ by $\theta$,

$$s = a \cos \theta.$$

Comparing with equation (4), it appears that we can thus construct for any position of $M$ the variable angle

$$\theta = nt.$$

The angle $AOP$ is thus directly proportional to the time; that is to say, while the motion of the particle at $M$ is governed by the force here considered, the point $P$ moves uniformly in the circumference of the circle. The constant velocity of $P$ is

$$a\frac{d\theta}{dt} = na = v_0;$$

that is (see Art. 335), it is the same as the velocity of $M$ when passing through the origin.

The motion is the vibratory motion known as ***harmonic motion***, which was used as an illustration of variable acceleration in Art. 11. It also occurs in Art. 44 as the projection of uniform circular motion. In the present notation, $n$ is the angular velocity of the circular motion.

## The Period of Harmonic Vibration.

338. Let $T$ denote the period of a *complete vibration* of $M$, that is, the interval of time in which it passes from $A$ to $B$ and back to $A$ again. This is the same as the time of a complete revolution of $P$, namely,

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\mu}}.$$

It is evident from equations (4) and (5) that for values of $t$ which differ by any multiple of this period the values of $s$ and $v$ are the same. At such times, the particle is said to be in ***the same phase*** of its vibration. The phase may be taken as the time which has elapsed since the particle was last at a given point moving in a given direction; thus the phase of a particle at $O$, Fig. 91, moving to the left is a quarter period in excess of that of a particle at $A$.

The expression for $T$ shows that the time of vibration is independent of $a$, which is called *the amplitude* of the vibration. The law of force with which we are here dealing is that which governs the vibrations of springs, and of stretched strings, and it is this fact which makes the musical tone produced by a tuning-fork or a stringed instrument independent of the distances through which the particles vibrate. The term *Harmonic Motion* is due to this connection with musical sounds.

### The Energy of Vibration.

**339.** It is shown in Art. 294 that the integral of the equation $vdv = f(s)ds$, when multiplied by $m$, is the equation of energy, expressing the equality of the work done with the kinetic energy gained in passing from one position to another. Thus, in the present case, equation (2), Art. 335, is equivalent to

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\mu a^2 - \tfrac{1}{2}m\mu s^2.$$

The terms in the second member are values of the potential function for this force taken, as in Art. 281, so as to vanish at the centre of force. Hence the equation expresses the work done in the form of loss of potential in passing from the distance $a$, where the body is at rest, to the distance $s$, where it has the kinetic energy $\tfrac{1}{2}mv^2$. When written in the form

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m\mu s^2 = \tfrac{1}{2}m\mu a^2,$$

it corresponds to the general equation $U + V = C$, Art. 279, and expresses the conservation of energy in its two mechanical forms. The total energy of vibration $\tfrac{1}{2}m\mu a^2$ varies as the square of the amplitude. It is all in the potential form at $A$ and at $B$, Fig. 91, and it is all in the kinetic form when the body is passing through $O$.

## Motion Produced by a Component of the Force.

**340.** It is a peculiarity of the law of force we are discussing that a resolved part of the force follows the same law, and therefore produces a motion of the same kind. Thus, suppose the particle $M$, Fig. 92, constrained to move in the smooth line $AB$, to be acted upon by a force, directed toward the fixed point $C$ (not in $AB$), and proportional to its distance from $C$, which we shall denote by $r$. Let $CO = b$ be the perpendicular from $C$ upon the line $AB$, and take $O$ as the origin of distance for the line $AB$. Then, $\mu$ denoting, as in Art. 335, the force acting on the unit mass at a unit's distance, the force acting upon $M$ in the direction $MC$ is $m\mu r$. The component in the direction $MO$ is therefore $m\mu r \cos OMC$; that is to say, $m\mu s$ acting toward the point $O$. Hence the force in the line of motion is precisely the same as that considered in Art. 335, and the motion is the same as if the particle were free, and $O$ were the centre of an attractive force proportional to the distance; that is to say, it is harmonic motion.

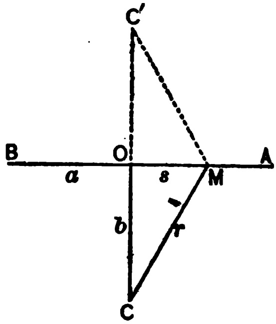


FIG. 92.

**341.** Let $C'$ be the point symmetrically situated to $C$ on the other side of the line $AB$; then, if the body were attracted to each of the points $C$ and $C'$ by forces each equal to $\frac{1}{2}m\mu r$, the force in $AB$ would be as before $m\mu s$, but there would be no force transverse to $AB$, so that the constraint of the line might be dispensed with, and the same harmonic motion would still exist.

If the force attracting the body to each of the points $C$ and $C'$ were constant and equal to $\frac{1}{2}m\mu b$, the force in $AB$ would be $m\mu b\frac{s}{r}$; and, since the fraction $\frac{b}{r}$, for small values of $s$, differs very

little from its maximum value unity, we shall have, for small oscillations, very nearly the same harmonic motion as for the force considered in the preceding articles. Again, whatever function of $r$ the force may be, if $P$ is its value at $O$, the attraction to $C$ and $C'$ will, for small oscillations, differ very little from $P$; hence the motion will still be very nearly harmonic. If $P$ is the tension of a light elastic string of length $CC' = l = 2b$, and $m$ is the mass of a bead at the middle point $O$, we shall have, by giving to $P$ the value assigned to the constant force above,

$$P = \tfrac{1}{2}m\mu b = \tfrac{1}{4}m\mu l, \qquad \text{whence} \qquad \mu = \frac{4P}{ml}.$$

Substituting this value of $\mu$ in the equation of Art. 338, we have

$$T = \pi\sqrt{\frac{ml}{P}}$$

for the period of small complete oscillations.*

## Repulsive Force Proportional to the Distance.

**342.** Let us now suppose the force to be repulsive and proportional to the distance from a fixed point in the line of motion,

* In the case of musical strings, the mass is uniformly distributed throughout the length; each particle has harmonic motion, but the period differs from that, when the mass is concentrated at the centre in the ratio of 2 to $\pi$. Thus,

$$T = 2\sqrt{\frac{ml}{P}},$$

and if $n$ is the number of vibrations per second and $w$ the weight of the string per linear unit (so that $mg = wl$), we have

$$n = \tfrac{1}{2}\sqrt{\frac{P}{ml}} = \frac{1}{2l}\sqrt{\frac{gP}{w}}.$$

Hence, the vibration number is inversely proportional to the length, and to the square root of the mass per linear unit, and directly proportional to the square root of the tension.

so that we have only to change the sign of $\mu$ in the differential equation of Art. 335; thus,

$$vdv = \mu s\,ds\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence, putting $\mu = n^2$ as before,

$$v^2 = n^2s^2 + C.$$

The second integration for this force takes different forms, according as this first constant is positive or negative, the cases corresponding to radically different kinds of motion. Thus, *if there is a position of rest*, we may take for initial values

$$s = a, \qquad v = 0, \qquad t = 0,$$

and the first integral equation becomes

$$v^2 = n^2(s^2 - a^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the constant of integration being negative. Substituting in $ds = vdt$, and separating the variables,

$$\frac{ds}{\sqrt{(s^2 - a^2)}} = ndt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Integrating,

$$\log[s + \sqrt{(s^2 - a^2)}] = nt + \log a, \quad . \quad . \quad . \quad (4)$$

the value of the constant being determined by the initial circumstances.

**343.** To express $s$ directly in terms of $t$, we have

$$\frac{s + \sqrt{(s^2 - a^2)}}{a} = e^{nt},$$

of which the reciprocal is

$$\frac{s - \sqrt{(s^2 - a^2)}}{a} = e^{-nt};$$

whence

$$s = \tfrac{1}{2}a(e^{nt} + e^{-nt}) = a \cosh nt. \quad . \quad . \quad . \quad . \quad (5)$$

Differentiating with respect to $t$,

$$v = \tfrac{1}{2}na(e^{nt} - e^{-nt}) = na \sinh nt. \quad . \quad . \quad . \quad . \quad (6)$$

In this case, the body recedes indefinitely from the centre of force, and when moving toward the centre $a$ is its least distance.

**344.** In the second case, the body may pass through the centre of force, and it is convenient to take for initial circumstances

$$s = 0, \qquad v = na, \qquad t = 0.$$

The first integral then takes the form

$$v^2 = n^2(s^2 + a^2);$$

whence

$$\frac{ds}{\sqrt{(s^2 + a^2)}} = nt.$$

Integrating, and determining the constant,

$$\log[s + \sqrt{(s^2 + a^2)}] = nt + \log a,$$

from which, proceeding as before,

$$s = \tfrac{1}{2}a(e^{nt} - e^{-nt}) = a \sinh nt.$$

Differentiating with respect to $t$,

$$v = \tfrac{1}{2}na(e^{nt} + e^{-nt}) = na \cosh nt.$$

**345.** There is also a special intermediate case in which the constant in the first integration is zero. The integral is then

$$v^2 = n^2 s^2,$$

whence

$$\frac{ds}{s} = n dt.$$

Integrating,

$$\log s = nt + \log b,$$

where $b$ is the value of $s$ when $t = 0$; therefore

$$s = be^{nt}; \qquad \text{whence} \qquad v = nbe^{nt}.$$

Values of $s$ less than $b$ here correspond to negative values of $t$, and $s = 0$ gives $t = -\infty$. If we reverse the direction of motion, we have $s = be^{-nt}$, $v = -nbe^{-nt}$, and $s = 0$ gives $t = \infty$; that is, the body will approach the origin indefinitely as a limiting position of rest.

## Attraction Inversely Proportional to the Square of the Distance.

**346.** Let us next consider the case of an attractive force varying inversely as the square of the distance $s$ from a fixed point taken as origin, so that, taking $s$ as positive, the acceleration is in the negative direction, and the equation of motion is

$$\frac{d^2 s}{dt^2} = -\frac{\mu}{s^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $s$ *must be positive*, because the second member does not change sign, as it should, when $s$ is negative. The integration of

$$v dv = -\mu \frac{ds}{s^2}$$

gives

$$v^2 = \frac{2\mu}{s} + C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As in the preceding case, the second integration takes different forms for values of $C$ differing in sign. In the first place, suppose that there is an instant (taken as the origin of time) for which the body is at rest, and let $a$ be the corresponding value of $s$, so that we have the initial values

$$v = 0, \qquad s = a, \qquad t = 0.$$

In this case, $C$ takes a negative value, and determining it by the initial values, equation (2) becomes

$$v^2 = 2\mu\left[\frac{1}{s} - \frac{1}{a}\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

whence

$$v = -\sqrt{\frac{2\mu}{a}}\sqrt{\frac{a-s}{s}},$$

the negative sign being taken because $v$ is negative for small *positive* values of $t$, that is to say, immediately *after* the instant of rest.

347. In this case, we have then to integrate

$$\frac{\sqrt{s}\,ds}{\sqrt{(a-s)}} = -\sqrt{\frac{2\mu}{a}}dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For this purpose, assume $\sqrt{s} = \sqrt{a}\cos\theta$, that is $s = a\cos^2\theta$; whence $\sqrt{(a-s)} = \sqrt{a}\sin\theta$, and $ds = -2a\cos\theta\sin\theta\,d\theta$. Substituting, the equation becomes

$$2a\cos^2\theta\,d\theta = \sqrt{\frac{2\mu}{a}}dt;$$

whence integrating, we have

$$a(\theta + \sin\theta\cos\theta) = \sqrt{\frac{2\mu}{a}}t,$$

the constant of integration vanishing because when $t = 0$, $s = a$

and therefore $\theta = 0$. Hence we obtain, for *the time of passing from rest at the distance a to the distance s*, the expression

$$t = \sqrt{\frac{a}{2\mu}}\left[a \cos^{-1}\sqrt{\frac{s}{a}} + \sqrt{(as - s^2)}\right], \quad . \quad . \quad (5)$$

in which the inverse cosine is the "primary value" of the function (Diff. Calc., Art. 55), because we have assumed in the preceding article that $t$ vanishes when $s = a$, and is positive while $s$ decreases from $a$ to zero.

Putting $s = 0$ in equation (5), we have, for the whole time of falling from the distance $a$ to the centre of force,

$$T = \frac{\pi}{\sqrt{\mu}}\left(\frac{a}{2}\right)^{\frac{3}{2}}.$$

348. In the second case, for which the constant of integration is positive, we may write equation (3) in the form

$$v^2 = 2\mu\left[\frac{1}{s} + \frac{1}{a}\right]; \quad . \quad . \quad . \quad . \quad . \quad . \quad (3')$$

whence, if we assume the motion to be from the centre,

$$v = \sqrt{\frac{2\mu}{a}}\sqrt{\frac{s+a}{s}}.$$

In this case, the velocity never vanishes, but decreases toward the limit $\sqrt{\frac{2\mu}{a}}$ corresponding to $s = \infty$.

We now have to integrate

$$\frac{\sqrt{s}\, ds}{\sqrt{(s+a)}} = \sqrt{\frac{2\mu}{a}}\, dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4')$$

for which purpose assume $\sqrt{s} = \sqrt{a} \tan\theta$, that is $s = a \tan^2\theta$, whence

$$\sqrt{(s + a)} = \sqrt{a} \sec\theta,$$

and

$$ds = 2a \tan\theta \sec^2\theta \, d\theta.$$

Thus equation (4′) becomes

$$2a \tan^2\theta \sec\theta \, d\theta = \sqrt{\frac{2\mu}{a}}\, dt.$$

Integrating, and determining the constant so that, when $t = 0$, $\theta = 0$ and therefore $s = 0$,

$$\sqrt{\frac{2\mu}{a}} \cdot t = a \tan\theta \sec\theta - a \log \frac{1 + \sin\theta}{\cos\theta};$$

whence

$$t = \sqrt{\frac{a}{2\mu}}\left[\sqrt{(s^2 + as)} - a \log \frac{\sqrt{(s + a)} + \sqrt{s}}{\sqrt{a}}\right]. \quad . \quad . \quad (5')$$

This is therefore the time of describing the distance $s$ from the centre, when the velocity is approaching the limiting value

$$\sqrt{\frac{2\mu}{a}}.$$

**349.** Finally, the integration takes still another form in the intermediate case when the constant in equation (2), Art. 346, is zero.

We now have

$$v^2 = \frac{2\mu}{s}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3'')$$

so that the velocity never vanishes, but has zero for its limit when $s = \infty$.

If we suppose the body to be receding from the centre of force,

$$\sqrt{s}\, ds = \sqrt{(2\mu)}\, dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4'')$$

Integrating, and supposing $\dot{s} = 0$ when $t = 0$, we have

$$\tfrac{2}{3}s^{\frac{3}{2}} = \sqrt{(2\mu)}\,t;$$

therefore

$$t = \frac{\sqrt{2}}{3\sqrt{\mu}}s^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5'')$$

is the time of describing $s$ from the centre when the velocity does not vanish for any finite distance but has zero for its limit. This value of $t$ is in fact the limiting value of that given in equation (5′) when $a$ is made infinite.

## The Gravitation Potential.

**350.** The law of force considered above is that of the attraction of gravitation when the attracting body is regarded as a fixed centre of force. The force acting on the mass $m$ is

$$F = -\frac{m\mu}{s^2}.$$

The potential function for this force is

$$U = -\int F ds = m\mu\int\frac{ds}{s^2} = -\frac{m\mu}{s} + C,$$

in which $\frac{m\mu}{s}$ is the value of the work-function $V$ (Art. 277). The constant $C$ may be so taken as to make the absolute value of the potential zero for any particular point. But, since there is no inconvenience in negative values of the potential, it is in general best to take $C = 0$, so that the potential (for positive values of $s$ only, see Art. 346) is defined by

$$U = -\frac{m\mu}{s},$$

which is zero at infinity and negative for all finite values of $s$.

351. Now, multiplying equation (3), Art. 346, by $\frac{1}{2}m$, it may be written

$$\tfrac{1}{2}mv^2 + U = -\frac{m\mu}{a}.$$

Hence, in the first of the above cases, the total energy is a negative constant equal to the value of the potential when the body is at rest. In the second case, that of Art. 348, equation (3′) gives, in like manner,

$$\tfrac{1}{2}mv^2 + U = \frac{m\mu}{a}.$$

Hence the total energy is in this case a positive quantity: The kinetic energy at every point exceeds the value of the negative potential, and the body may recede to infinity with a limiting velocity

$$v = \sqrt{\frac{2\mu}{a}}.$$

In the third case, the total energy is zero, the kinetic energy having at every point a value numerically equal to the potential energy.

352. The same law of inverse squares holds for the attraction to the centre of a sphere composed of matter either of uniform density, or of variable density which has the same value at points equally distant from the centre. Assuming the earth to be such a sphere of radius $R$, it is convenient to take the potential so as to vanish at the surface. We therefore add to the expression for $U$ the positive constant $\frac{m\mu}{R}$, equal to the negative value assigned in Art. 350 to $U$ at the surface, and thus obtain

$$U = m\mu\left[\frac{1}{R} - \frac{1}{R+h}\right] = \frac{m\mu h}{R(R+h)}$$

for the potential at the distance from the centre $s = R + h$; that is, at the height $h$ above the surface.

The acceleration at the surface is $g = \dfrac{\mu}{R^2}$: hence, substituting $\mu = gR^2$, we have

$$U = mgh\frac{R}{R+h} = Wh\,\frac{R}{R+h}.$$

When $h$ is small compared with $R$, this is nearly equivalent to $Wh$, which is the potential energy at the height $h$ when the force $W$ is regarded as constant, and the potential is so taken as to vanish at the surface.

Putting $h = \infty$, the expression above gives $mgR$ for the potential at an infinite distance. Accordingly, this is also the value of the kinetic energy of a body falling with no initial velocity from an infinite distance to the surface of the earth. Putting $mgR = \frac{1}{2}mv^2$, we have for the corresponding velocity

$$v = \sqrt{(2gR)}.$$

The numerical value of this velocity is about 7 miles per second; hence, if a body were projected upward with such a velocity and free from any other resistance, it would escape from the sphere of the earth's attraction.

### EXAMPLES. XVIII.

1. A particle moves in a straight line subject to an attraction proportional to $s^{-\frac{3}{2}}$. Show that the velocity acquired in falling from an infinite distance to the distance $a$ is equal to that acquired in falling from rest at $a$ to a distance $\frac{1}{4}a$.

2. If a particle be attracted to two centres of force proportional to the distance with intensities $\mu_1$ and $\mu_2$, show that the resultant at all points is directed toward the weighted centre of gravity of the given centres, is proportional to the distance and has an intensity $\mu_1 + \mu_2$ at a unit's distance; and hence that the motion will be harmonic. Show also that the property extends to any number of centres of force proportional to the distance. (See Art. 67.)

3. Let a weight $W$ hang from an elastic string without weight, stretching the string to a length exceeding its natural length by $e$. Assuming Hooke's Law, show that, if the weight be now drawn down a further distance less than $e$ and then released, its vertical motion will be harmonic, and find the time of a complete vibration.

$$2\pi\sqrt{\frac{e}{g}}.$$

4. Let a light elastic string be stretched to an additional length $e$, the tension being $P$, and let it carry a bead of mass $m$ at its middle point. Show that the motion of the bead, when displaced in the direction of the string and released, is harmonic, and find the time of a complete vibration.

$$\pi\sqrt{\frac{em}{P}}.$$

5. A heavy body, attached to a fixed point by an elastic string whose natural length is $a$, hangs freely, stretching the string to an additional length $e$. It is drawn down through a further distance $c > e$ and released. Determine the distance through which it will rise if $c^2 < e^2 + 4ae$.

$$\frac{(e+c)^2}{2e}.$$

6. If, in the preceding example, $c^2 > 4ae + e^2$, show that the body will rise until the string is stretched vertically upward to the length $a + x$, where $x$ is determined by the equation

$$(x + e)^2 = c^2 - 4ae.$$

7. Find the time it takes the back-weight in Ex. 6, XIV., to rise, when the hand-lever is suddenly pulled back and locked. Find also its final velocity, and verify that its final kinetic energy is equivalent to the extra work done in the sudden motion.

$$t = \frac{\pi}{2}\sqrt{\frac{3}{2g}}; \quad \frac{1}{6}\sqrt{\frac{2g}{3}}.$$

8. A particle of unit mass is attached by a straight elastic string to a centre of repulsive force equal to $\mu$ times the distance; the string is at first of its natural length $a$, and its tension when stretched one unit is $k$. Supposing $\mu < k$, find the greatest

distance from the centre of force which the body will reach, and the time it will take to return to its first position.

$$\frac{k+\mu}{k-\mu}a; \quad \frac{2\pi}{\sqrt{(k-\mu)}}.$$

9. A perfectly flexible rope whose weight is $w$ per linear unit and length $2l$ rests in equilibrium on a smooth peg. If now one end be raised a distance $a$ and then released, find the time in which this end will rise to the height $x$ above its original position, and the tension at that instant of the rope at the point where it passes over the peg.

$$\sqrt{\frac{l}{g}}\log\frac{x+\sqrt{(x^2-a^2)}}{a}; \quad w\frac{l^2-x^2}{l}.$$

10. If, when in equilibrium, the rope in the preceding example had been given an initial velocity $v_0$, how long would it take to drop from the peg?

$$\sqrt{\frac{l}{g}}\log\frac{\sqrt{(lg)}+\sqrt{(lg+v_0^2)}}{v_0}.$$

11. In the case of a force inversely proportional to the square of the distance, if $f_0$, $s_0$, $v_0$ denote the acceleration, distance and velocity at any point, show that the motion belongs to the case considered in Art. 347, Art. 348 or Art. 349 according as $v_0^2$ is less than, greater than or equal to $2f_0s_0$.

12. Show that the time of descent from rest through the first half of the distance to a centre of attraction varying as $(\text{distance})^{-2}$ is to that through the last half as $\pi+2:\pi-2$.

13. If $h$ be the height due to a given velocity at the earth's surface, supposing the attraction constant (see Art. 297), and $H$ the corresponding height, when the variation of gravity is taken into account, prove that

$$\frac{1}{H}=\frac{1}{h}-\frac{1}{R}.$$

14. Assuming the attraction of a sphere upon a particle to be the same as that of the entire mass supposed concentrated at the centre, show that, if a sphere of the same density as the earth attract a free particle placed at a distance from its surface bearing

a given ratio to the radius, the time of falling to the surface will be the same as that of a particle falling to the earth's surface from a distance bearing the same ratio to the earth's radius.

15. If the intensity of an attractive force be $\frac{\mu}{s^n}$, show that, when $n > 1$, the velocity acquired by falling from an infinite distance to the distance $a$ is

$$v = \sqrt{\frac{2\mu}{n-1}a^{-\frac{1}{2}(n-1)}},$$

and that, if the potential is so taken as to vanish at infinity, its value at $a$ is the negative of the corresponding kinetic energy.

16. In the preceding example, if $n < 1$, show that the velocity acquired in falling from rest at the distance $a$ to the centre of force is

$$v = \sqrt{\frac{2\mu}{1-n}a^{\frac{1}{2}(1-n)}},$$

and that the potential may be so taken as to vanish at the centre; its value at the distance $a$ being then the kinetic energy corresponding to this value of $v$.

17. If, in Ex. 15, $n = 1$, show that the potential is infinite both at infinity and at the centre, and, if so taken as to vanish at $s = a$, is

$$U = m\mu \log \frac{s}{a}.$$

Also, being given that $\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}$, find the time of falling from the distance $a$ to the centre of force.

$$a\sqrt{\frac{\pi}{2\mu}}.$$

18. A ship is rolling through the angle $2\phi$ from extreme port to extreme starboard in $d$ seconds. Assuming the motion to be one of harmonic oscillation in a circular arc about an axis in the

water-line, find the force required to prevent a weight $W$ from slipping upon a deck $b$ feet above the water-line.

$$\left(\sin\phi + \frac{\pi^2 b\phi}{gd^2}\right)W.$$

19. A weight capable of stretching the spring of a spring balance $1\frac{1}{2}$ inches is dropped upon the scale-pan from a height of 6 inches. Neglecting the mass of the scale-pan, find the time after the instant of striking when the body will come to rest.

0.119 sec.

## XIX.

## Curvilinear Motion.

**353.** When a particle is moving in a given curve in a perfectly defined manner, so that the hodograph (see Art. 37) of the motion could be drawn, it has at each point a definite acceleration, which is graphically illustrated, when the hodograph is drawn, by the velocity of the auxiliary point in the hodograph, Art. 39. Thus, in the case illustrated in Fig. 5, if at $P$ we construct a vector representing in direction and magnitude the velocity of $P'$ in the hodograph, it will represent the acceleration which actually takes place in the supposed motion of $P$.

The vector so constructed gives the magnitude and direction of the single force which would account for the supposed motion in a free particle of mass unity. The inertia of the particle is a force equal and directly opposite to this force.

### Tangential and Normal Components of Acceleration.

**354.** The acceleration which actually takes place in any given motion may be resolved into components in a variety of ways; and, by the Second Law of Motion, these correspond to

the like methods of resolving the force or system of forces which produces the motion. In Art. 42, this is done for a plane motion in two fixed directions at right angles to one another. This method corresponds to the resolution of forces in the direction of fixed axes, and is the most convenient in finding the equation of the curve, as for example in Art. 324.

355. It is useful for some purposes to resolve the acceleration into components along the tangent and normal to the curve. Denoting, as usual, by $v$ the numerical value of the velocity in the curve, that is, the speed, it is obvious that the tangential component of the acceleration is the rate of change in the speed, that is

$$\frac{dv}{dt};$$

since the normal component, being perpendicular to the path, cannot hasten or retard the body in its path.

It follows that, if the velocity in the curve is constant, there is no tangential acceleration, that is to say, the acceleration is entirely normal. We have already seen in Art. 40 that this is the case in uniform circular motion, and the value of the acceleration was found in that case to be $\frac{v^2}{a}$, where $a$ is the radius of the circle.

## The Normal Component of Inertia.

356. The inertia of the particle of unit mass is, in like manner, resolved into two components which are equal and opposite to the tangential and normal components of the acceleration. Thus, while the tangential inertia resists any change of the velocity in the curve (exactly as the whole inertia does in rectilinear motion), the normal inertia resists the deflection of the path from the straight line in which it would move if it were

not acted upon by a force transverse to its path. Thus in Fig. 89 the acceleration is normal at the point $O$, and the inertia which acts upward is, at that point, simply the resistance of the body to being moved away from the tangent at $O$. At any other point the inertia, which acts vertically upward, has a component which resists the change of velocity in the curve as well as one which resists the deflection of the path.

## Centrifugal Force.

357. We have already seen in Art. 40 that, in the case of uniform circular motion, the acceleration is purely normal, being directed always toward the centre, and that it has the constant value

$$f = \frac{v^2}{a},$$

where $v$ is the constant linear velocity and $a$ is the radius of the circle. It follows that the only force necessary to keep a particle of mass $m$ moving uniformly in a circle is

$$F = \frac{mv^2}{a}.$$

This may be supplied by the tension of a string connecting the particle with the fixed centre; and, in order that it may act freely, we may conceive the motion to take place on a smooth horizontal table, so that the resistance of the table neutralizes the weight of the particle.

The force $F$ which produces the acceleration is called *the centripetal force* because it is directed toward the centre, and the inertia of the particle which is equal and opposite to this force is called *the centrifugal force*. It is, in this case, the

inertia force directed away from the centre which renders the centripetal force $F$ necessary; and it is to be noticed that the force $F$, being always perpendicular to the direction of displacement, does no work. Thus no work is done, in this case, against inertia; accordingly no change takes place in the kinetic energy.

358. If $\omega$ is the angular velocity of the particle $m$ in the circle whose radius is $a$, the linear velocity is $v = a\omega$. Making this substitution, we have the expression

$$f = a\omega^2$$

for the acceleration, or force acting on a unit mass. Referring to Fig. 91, Art. 335, we notice that, if $P$ is moving with the angular velocity $n$, the resolved part of this force along the axis of $x$ is $-n^2x$. This is exactly the force which, in that article, we have supposed to act upon the particle at $M$, and which, as we have seen, produces the harmonic motion which is the projection of the motion of $P$. Thus we may regard the two rectangular components of the centripetal force, as independently producing the two harmonic motions of which the circular motion is the resultant.

359. If $T$ is the time in seconds of a complete revolution we have $T\omega = 2\pi$, and if $n$ is the number of revolutions per second, $nT = 1$; using these quantities to replace $v$ or $\omega$, and also putting $\frac{W}{g}$ for $m$, we have the following expression for centrifugal force:

$$F = \frac{Wv^2}{ga} = \frac{Wa\omega^2}{g} = \frac{4\pi^2 aW}{gT^2} = \frac{4n^2\pi^2 aW}{g}.$$

These expressions show that, while the centrifugal force is *inversely* proportional to the radius for a given *linear* velocity, it is for a given *angular* velocity, or for a given time of revolution, *directly* proportional to the radius.

## The Conical Pendulum.

**360.** In the case of the heavy particle moving in a horizontal circle, the resultant of the weight (which in Art. 357 we supposed neutralized by the resistance of a smooth horizontal plane) and the centrifugal force will lie in the vertical plane which passes through the radius of the horizontal circle. In Fig. 93 let this resultant meet the vertical line through the centre in $C$, and denote $CO$ by $h$. The triangle $OCM$ will serve as a triangle of forces, and gives

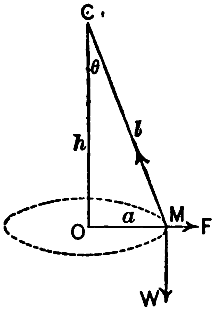


FIG. 93.

$$\frac{a}{h} = \frac{F}{W} = \frac{4\pi^2 a}{gT^2};$$

whence

$$h = \frac{gT^2}{4\pi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus, for a given time of revolution, the height $h$ is independent not only of the weight, but of the radius of the circle described.

**361.** The forces of constraint may now be completely replaced by the tension of a string or rod connecting the particle $M$ with $C$. This string will, in the revolution, describe the surface of a right cone. For this reason, the arrangement is called *the conical pendulum.*

Denoting the length $CM$ by $l$, and the angle $OCM$ by $\theta$, we have $h = l \cos \theta$. Also, in the case of rapid motion, let $n$ be the number of revolutions per second, so that $nT = 1$. Then, from equation, (1) we have

$$\cos \theta = \frac{gT^2}{4l\pi^2} = \frac{g}{4n^2 l\pi^2}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which determines $\theta$ for a given rapidity of motion. The tension of the string is then $W \sec \theta$.

The principle of the conical pendulum is employed for the

regulation of the speed of a shaft in the "governor" of the steam-engine, in which the increase of the angle $\theta$ beyond the desired limit is made to operate a valve cutting off steam. It is also used in the clockwork for driving a telescope equatorially mounted, in which case the increase of $\theta$ causes sufficient friction of the body $M$ against a metal ring (whose inner surface is the desired circle of revolution) to produce the necessary retardation.

## The Centrifugal Force due to the Earth's Rotation.

362. Let $NQS$, Fig. 94, be a section of the earth supposed a sphere, and $Q$ a point on the equator. By Art. 359, the centrifugal force acting on a unit mass at $Q$ by virtue of the earth's rotation is

$$f = \frac{4\pi^2 R}{T^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $T$ is the number of seconds in the sidereal day, and $R$ the number of feet in the radius of the earth. The value of $f$ is thus found to be 0.1113, which is about $\frac{1}{288}$ of the observed value, namely 32.09, of $g$ at the equator. This force tends directly to diminish the weight of the body. Hence, denoting by $G$ the value which $g$ would have if there were no rotation,

$$G = g + f = 32.20,$$

and the centrifugal force diminishes the weight of a body at the equator by about $\frac{1}{289}$.

363. Let $P$ be a body at a place whose latitude is $\lambda$, and draw $PD$ perpendicular to the axis; then $P$ describes a circle whose radius is $PD = R \cos \lambda$. Hence, by Art. 359, the centrifugal force on a unit of mass at $P$ is $\frac{4\pi^2 R \cos \lambda}{T^2}$, which by equation (1) reduces to

$$f \cos \lambda.$$

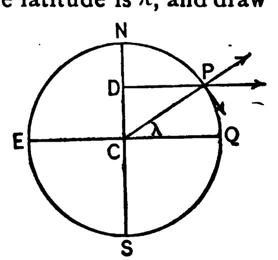

FIG. 94.

This force acts in the direction $DP$ as represented in the figure. Resolving it into rectangular com-

ponents along, and perpendicular to, the earth's radius, they are

$$f \cos^2 \lambda, \qquad \text{and} \qquad f \cos \lambda \sin \lambda.$$

The first of these, which tends directly to diminish the weight, decreases with the increase of latitude, and causes an increase in the value of $g$ as we approach the poles. The second produces a deflection in the direction of gravity, which is in fact the resultant of the earth's attraction and the centrifugal force. Since the sea-level is everywhere perpendicular to this resultant, the centrifugal force causes the earth to assume a form of equilibrium which has been proved to be a spheroid, the polar diameter being smaller than the equatorial. This, in accordance with the law of gravitation, still further increases the difference between the values of $g$ at the equator and the poles.

## The General Expression for the Normal Acceleration.

**364.** The hodograph may be used to find the expression for the normal component of acceleration in the general case as well as in that of uniform circular motion. In Fig. 95, the right-hand diagram represents the curve in which the particle $P$ moves, $\phi$ denoting the inclination of the tangent to a fixed line. The left-hand diagram is the hodograph, which we refer to polar coordinates, the initial line being in the direction $\phi = 0$. Then,

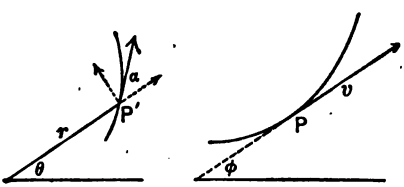


FIG. 95.

by the construction of the hodograph, the polar coordinates of $P'$, the point corresponding to $P$, are

$$r = v, \qquad \theta = \phi.$$

We have seen that the acceleration $\alpha$ of $P$ is the same as the velocity of $P'$; hence the tangential and normal components of the acceleration are the resolved parts of the velocity of $P'$ in

the direction of, and perpendicular to, the radius-vector $r$. Hence (Diff. Calc., Art. 317) they are

$$\frac{dr}{dt} = \frac{dv}{dt}, \qquad \text{and} \qquad \frac{rd\theta}{dt} = \frac{vd\phi}{dt}.$$

The first of these expressions is the value of the tangential acceleration already given in Art. 355.

The normal acceleration is more conveniently expressed in terms of the radius of curvature at the point $P$, which is (Diff Calc., Art. 332)

$$\rho = \frac{ds}{d\phi}; \qquad \text{whence} \qquad \frac{d\phi}{dt} = \frac{ds}{\rho dt} = \frac{v}{\rho}.$$

Substituting in the expression above, we have

$$\text{Normal acceleration} = \frac{v^2}{\rho}.$$

The result found in Art. 90, for the special case in which $v$ and $a$ are constants, agrees with this general expression.

365. If the body is constrained to move in a smooth fixed curve, and there is no external force acting upon it except the reaction of the curve, there will be no tangential acceleration, and therefore $v$ will remain constant. The pressure on the curve caused by the normal inertia, or centrifugal force, will now be

$$\frac{mv^2}{\rho}, \qquad \text{or} \qquad \frac{Wv^2}{g\rho}.$$

If the curve is horizontal and the weight of the body is also regarded as acting, this is, of course, only the horizontal component of the action between the curve and the body.

If the curve is rough, the friction caused by this pressure will be a tangential force causing a retardation. The equation of the motion will therefore be

$$\frac{dv}{dt} = -\mu\frac{v^2}{\rho},$$

which is directly integrable when $\rho$ is constant. See Ex. 15.

## EXAMPLES. XIX.

1. A cord two feet long passes at its middle point through a hole in a smooth horizontal table. It carries at its lower end a weight of two pounds, and at the other a weight of one pound. With what velocity must the latter weight revolve in a circle to prevent the lower weight from descending?

$v = \sqrt{(2g)} = 8 \text{ f}/\text{s}.$

2. If, in the preceding example, only $\frac{1}{4}$ of the cord lies on the table, how many revolutions must be made per minute to sustain the weight? 108.

3. With what number of turns per minute must a weight of 10 grammes revolve on a smooth horizontal table, at the end of a string half a meter in length, to cause the same tension that would be caused by a weight of one gramme hanging vertically at a place where the value of $g$ in meters is 9.81? About 13.4.

4. A weight of $W$ pounds is connected by a string of length $a$ to a fixed point of a smooth horizontal table; the string can only support a weight of $W_1$ pounds. What is the greatest number of revolutions per second which $W$ can make without breaking the string?

$$n = \frac{1}{2\pi}\sqrt{\frac{W_1 g}{Wa}}.$$

5. A string can just carry one pound. What is the shortest length of this string which can connect a bullet weighing one ounce and moving with a velocity of 40 feet per second to a fixed point? $3\frac{1}{8}$ feet.

6. If the masses of the bodies in Ex. II. 24 are $m$ and $m'$, the length of the string $l$, and the angular velocity $\omega$, show that the tension of the string is

$$\frac{mm'l}{m + m'}\,\omega^2;$$

and find its value, supposing the bodies to weigh respectively 1 and 5 pounds, the string to be 3 feet long, and 200 revolutions to be made per minute. 34.32 pounds.

7. A stone weighing one pound is whirled round by means of a string so as to describe a horizontal circle in a plane 2 feet be-

low the point of suspension. Find the time of revolution and also the tension, $l$ being the number of feet in the length of the string.

$2\pi\sqrt{\frac{2}{g}}$ sec.; $\frac{1}{2}l$ pounds.

8. A railway curve has a radius of a quarter of a mile, and trains are to run over it at the rate of 20 miles an hour, the gauge being 4 ft. 8 in. How much should the outer rail be raised above the level of the inner one to prevent lateral pressure on the rails?

About $1\frac{1}{8}$ in.

9. A particle rests in equilibrium at any point of a bowl in the form of a solid of revolution rotating once in $T$ seconds about its axis, which is vertical. Show that the form is that of the paraboloid whose latus rectum is $\frac{gT^2}{2\pi^2}$.

10. The length, weight and period of a conical pendulum being given, show that the tension of the string is independent of the value of $g$.

11. A weight attached to a fixed point by a string describes a horizontal circle, the string being inclined 60° to the vertical. Show that the velocity is equal to that due to a height equal to three-fourths of the length of the string.

12. A plummet is suspended from the roof of a railway car. How much will it be deflected from the vertical when the train is running 45 miles per hour over a curve of 300 yards radius?

$8°\,36'$.

13. Assuming $g = 32$, and the earth's radius 4000 miles, in what time could a body revolve freely round the earth close to its surface? $1^h\ 25^m\ 4^s$.

14. Supposing the earth a sphere of 4000 miles radius, find approximately the greatest value of the deviation of gravity from the direction of the radius. $6'$.

15. A particle without weight is projected tangentially with the velocity $v_0$ into a rough circular tube of radius $a$, $\mu$ being the coefficient of friction. Show that the space described in the time $t$ is

$$s = \frac{a}{\mu}\log\frac{a + \mu v_0 t}{a}.$$

Show also that the times in which successive revolutions are made are in geometrical progression, and that, when the particle has the velocity $v$, it cannot have been moving more than $\dfrac{2}{\mu v}$ seconds.

16. Show that the hodograph of the motion in the preceding example is the logarithmic spiral

$$r = v_0 e^{-\mu\theta}.$$

17. A smooth tube rotates with uniform angular velocity $\omega$ about a vertical axis intersecting it at right angles. A particle in the tube at the distance $a$ from the axis is released Show that its distance $r$ at the end of the time $t$ is

$$r = \tfrac{1}{2}a(e^{\omega t} + e^{-\omega t}) = a \cosh \omega t,$$

so that the polar equation of its path is $r = a \cosh \theta$.

## XX.

## Constrained Motion under the Action of External Force.

366. When a body acted upon by a force is constrained to move in a smooth curvilinear path, the tangential component of the force is resisted only by the corresponding component of the body's inertia. The change of velocity is therefore determined solely by this component of the force, exactly as in the case of rectilinear motion. That is to say, $v$ is determined by the integration of

$$vdv = fds,$$

where $f$ is the *tangential* force acting on a unit mass expressed in terms of the distance measured along the arc from some fixed point. The position of the body at any time $t$ may then be determined by the integration of $vdt = ds$.

367. Resolving forces normally to the curve, we have a con-

dition of kinetic equilibrium which involves three forces, namely, the resistance of the curve, the normal component of the external force, and the centrifugal force, or normal component of inertia. This last is given by the expression

$$\frac{mv^2}{\rho}$$

(Art. 365) after $v$ has been determined, as explained in the preceding article, by means of the tangential force. It follows that the resistance necessary to keep the body in the given path may reverse its direction, and if the body moves on the surface of a fixed solid, so that it is free to leave the curve on one side, it will do so at the point where the resistance of the surface vanishes; that is, where the normal component of the force is equal and opposite to that of inertia.

It is obvious that at such a point the curve of constraint has the same curvature as the free path in which the body subsequently moves.

## Motion of a Heavy Body on a Smooth Vertical Curve.

368. In the case of a body sliding down a smooth curve in a vertical plane, let us refer the curve to rectangular axes, that of $x$ being horizontal.

Let $M$, Fig. 96, be the position of the body whose mass is $m$ and weight $W$; then, $\phi$ denoting the inclination of the tangent to the axis of $x$, $W \sin \phi$ is the tangential force and

$$f = g \sin \phi,$$

the acceleration down the curve.

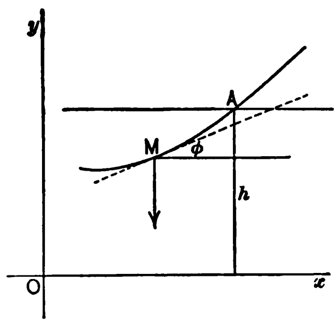

Fig. 96.

Hence, if $s$ is measured as usual in the direction determined by

the angle $\phi$ (that is, up the curve in the diagram), the equation of motion is

$$vdv = -g \sin \phi \, ds$$

or, since $\sin \phi = \frac{dy}{ds}$,

$$vdv = -gdy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $A$ be the point at which the body is at rest, and let $h$ be the ordinate of $A$; then, integrating, and determining the constant in such a way that $v = 0$ when $y = h$, we have

$$v^2 = 2g(h - y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

**369.** Introducing the factor $m$, equation (2) may be written

$$\tfrac{1}{2}mv^2 + Wy = Wh.$$

Regarding the potential energy of a body as zero upon the axis of $x$, this expresses that the sum of the kinetic and potential energies is constantly equal to the initial energy, which is all in potential form at $A$. The motion will be continuous until the body reaches a point on the same level with $A$, in which case it will come to rest, and then, unless the tangent at that point is horizontal, it will return and again come to rest at the point $A$.

The velocity at any point is, by equation (2), that due to the distance of the point below the level of $A$. Hence, if the velocity at any point is known, this level may be constructed even when the curve lies entirely below it. It is sometimes called *the level of zero-velocity*, and corresponds to the directrix in the case of free parabolic motion. (See Art. 321.)

**370.** Supposing the body to move on the surface of a solid, the centrifugal force will, in any position of the curve which is *convex* as viewed from above, diminish the pressure upon the surface, and the body will leave the curve when the centrifugal force be-

comes equal to $W \cos \phi$, the normal component of the weight; that is, when

$$W \cos \phi = \frac{mv^2}{\rho}.$$

Substituting the value of $v^2$ found in Art. 368, this equation becomes

$$\tfrac{1}{2}\rho \cos \phi = h - y. \quad \ldots \ldots \quad (1)$$

If we draw the circle of curvature, which in this case will lie below the curve, it is easily seen that $2\rho \cos \phi$ is the vertical chord through $M$ of this circle. Hence, by equation (1), the point at which the body will leave the curve is that at which the vertical chord of curvature is four times the distance of the point below the level of zero-velocity; or, what is the same thing, the point for which the centre of curvature is three times as far as the point itself is from this level.

The parabola in which the body subsequently moves is that of which the level of zero-velocity is the directrix. It follows that the centre of curvature for any point of the parabola is three times as far as the point itself is from the directrix.

## The Cycloidal Pendulum.

371. To determine the position of the body moving in a smooth vertical curve at a given time, it is necessary (see Art. 366) to integrate the equation $ds = v dt$, which, by equation (2), Art. 368, is in this case

$$\frac{ds}{\sqrt{(h-y)}} = \sqrt{(2g)}\, dt.$$

To integrate this $y$ must be expressed as a function of $s$ (or $y$ and $ds$ in terms of some other variable) by means of the equation of the curve.

For example, suppose the curve to be an inverted cycloid. The equations of the curve, the vertex being at the origin, are (Diff. Calc., Art. 290)

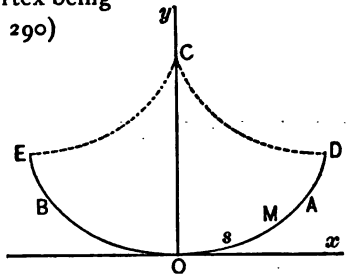


FIG. 97.

$$\left.\begin{aligned} x &= a(\psi + \sin\psi), \\ y &= a(1 - \cos\psi) \\ &= 2a\sin^2\tfrac{1}{2}\psi; \end{aligned}\right\} \quad . \quad . \quad (1)$$

whence

$$\left.\begin{aligned} dx &= a(1 + \cos\psi)d\psi, \\ dy &= a\sin\psi\, d\psi, \end{aligned}\right\} . \quad (2)$$

and, from $ds = \sqrt{(dx^2 + dy^2)}$,

$$ds = 2a\cos\tfrac{1}{2}\psi\, d\psi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Hence, if $s$ is measured from the origin, we have, by integration,

$$s = 4a\sin\tfrac{1}{2}\psi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and therefore, since $y = 2a\sin^2\tfrac{1}{2}\psi$,

$$y = \frac{s^2}{8a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting this value of $y$ in terms of $s$, the differential equation above becomes

$$\sqrt{(2g)}dt = \frac{\sqrt{(8a)}}{\sqrt{(8ah - s^2)}}ds,$$

or

$$\sqrt{\frac{g}{4a}}\,dt = \frac{ds}{\sqrt{(8ah - s^2)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which $h$ stands for the ordinate of the point $A$ where the body is assumed to start from rest. This equation is of the same form as equation (3), Art. 336, the value of $\mu$ being $\dfrac{g}{4a}$. Hence the

motion as measured along the arc is harmonic; and, by Art. 338, the time occupied in passing from $A$ to $B$ and returning to $A$ is

$$T = \frac{2\pi}{\sqrt{\mu}} = 4\pi\sqrt{\frac{a}{g}}.$$

Since this is independent of the position of $A$, the time of vibration is the same for all arcs of the cycloid; the curve is for this reason sometimes called *the Tautochrone.*

**372.** In the equations above, $a$ stands for the radius of the generating circle of the cycloid. The evolute of the cycloid is an equal cycloid, having its vertices at the cusps of the given cycloid as represented in Fig. 97 (Diff. Calc., Art. 357). Hence, if a heavy particle be suspended from $C$ by a string of length $4a$, and in its vibration be made to wrap upon solid pieces having the form of the cycloidal arcs $CD$, $CE$, it will describe the cycloid $AOB$. Such an arrangement is called a *Cycloidal Pendulum.* Putting $l = 4a$ for the length of the string, and $\tau = \frac{1}{2}T$, we have

$$\tau = \pi\sqrt{\frac{l}{g}}$$

for the time of passing from $A$ to $B$, which is the time of a single swing, or *beat*, of the cycloidal pendulum.

If the cycloidal pieces be removed we have the *simple pendulum*, the particle describing the circle of curvature of the cycloid at the vertex. It is hence evident that for small oscillations $\tau$ is very nearly the period of the beats of the simple pendulum.

**373.** We found in Art. 337 that harmonic motion resulted from an attractive force proportional to the distance measured from a fixed point of the path. Accordingly, the harmonic motion, in this case, results from the fact that the tangential force (which alone produces the motion) is proportional to $s$, the distance measured along the path, and acts toward $O$. For this force is $f = -g \sin\phi$, and by equations (2), (3) and (4), Art. 371,

$$\sin\phi = \frac{dy}{ds} = \sin\tfrac{1}{2}\psi = \frac{s}{4a}; \quad \text{hence} \quad f = -\frac{gs}{4a}.$$

## Motion in a Vertical Circle.

**374.** Let $C$, Fig. 98, be the centre and $a$ the radius of a circle in a vertical plane. Take $O$, the lowest point, as the origin; let $V$ be the velocity at $O$ of a heavy particle moving smoothly in the circle, and $\theta$ the angle $OCM$ which defines the position of the particle $M$ at the time $t$ reckoned from the instant when the particle was at $O$. The acceleration down the curve is $g \sin \theta$ acting in the opposite direction to that in which $s$ is measured. Hence the equation of motion is

$$\frac{d^2s}{dt^2} = -g \sin \theta.$$

This is equivalent to $vdv = -g \sin \theta \, ds$, and the first integration gives, as in Art. 368 (since $\phi = \theta$),

$$v^2 = 2g(h - y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The constant of integration $h$ is the height due to the velocity when $y = 0$; that is,

$$h = \frac{V^2}{2g}.$$

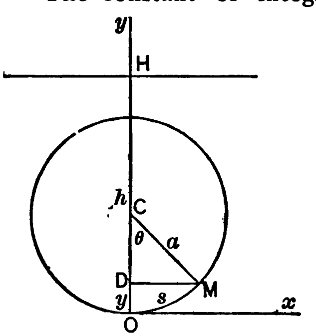

FIG. 98.

Let $OH$, Fig. 98, be this height, and draw the level of "zero-velocity." In the diagram we have assumed $h > 2a$ the diameter of the circle, so that $M$ will, in this case, move continuously around the circle.

**375.** Since $s = a\theta$, $ds = ad\theta$, and equation (1) gives for the second integration

$$v = \frac{ad\theta}{dt} = \sqrt{(2g)}\sqrt{(h - y)};$$

whence

$$\frac{\sqrt{(2g)}}{a} dt = \frac{d\theta}{\sqrt{(h - y)}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which $y$ is to be expressed in terms of $\theta$. Draw $MD$ perpendicular to $CO$, then

$$y = OD = a - a\cos\theta,$$

or

$$y = 2a\sin^2\tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Making this substitution, and putting $\psi$ for $\frac{1}{2}\theta$, equation (2) may be written

$$\frac{\sqrt{(2gh)}}{a}dt = \frac{2d\psi}{\sqrt{\left(1 - \dfrac{2a}{h}\sin^2\psi\right)}},$$

or, putting $\dfrac{2a}{h} = \kappa^2$, and integrating,

$$\frac{V}{2a}t = \int_0 \frac{d\psi}{\sqrt{(1 - \kappa^2\sin^2\psi)}}, \quad . \quad . \quad . \quad (4)$$

the lower limits being the value of $\psi$ which corresponds to $t = 0$. The integral is Legendre's *Elliptic Integral* of the first kind, which he denoted by $F(\psi, \kappa)$. $\psi$ is called the amplitude, and $\kappa$ the modulus of $F(\psi, \kappa)$. Legendre considered the integral to be in its standard form when $\kappa$ is less than unity (as it is in the present case), and for this form he published tables of its numerical values. (Legendre's *Fonctions Elliptique*, Vol. II.)

376. As mentioned in Art. 374 the particle in this case makes complete revolutions in the circle to which we may imagine its motion to be constrained by means of a rod of length $a$ connecting it with the fixed point $C$. Denoting by $T$ the time of a complete revolution, $\frac{1}{2}T$ is the value of $t$ when $y = 2a$; that is, by equation (3), when $\psi = \frac{1}{2}\pi$. Therefore

$$\frac{V}{4a}T = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{(1 - \kappa^2\sin^2\psi)}} = K. \quad . \quad . \quad . \quad . \quad (5)$$

This definite integral, which is $F(\frac{1}{2}\pi, \kappa)$, is called *the complete elliptic integral*, and is usually denoted by $K$. Separate tables of

the values of $K$ for different values of $\kappa$ are given by Legendre, and also in Bertrand's *Calcul Integral*, p. 714, Greenhill's *Elliptic Functions*, p. 10, etc.

When $\kappa = 0$, the value of $K$, see equation (5), reduces to $\frac{1}{2}\pi$. Hence, when $h$ increases indefinitely so that the limit of $\kappa$ is zero, equation (5) gives $VT = 2a\pi$, as it should, since in the limit the velocity is evidently constant.

**377.** Let us next suppose $h < 2a$, then $v$ in equation (1) vanishes when $y = h$. The level of zero-velocity, cutting off $OH = h$ on the vertical diameter, will now cut the circle in $A$ and $B$, Fig. 99. The particle will come to rest at $A$ and the motion will be one of oscillation between $A$ and $B$. The relation between $t$ and $\theta$ or $2\psi$ is still expressed by equation (4), but the value of $\kappa$ will be greater than unity, so that the integral will not be of the standard form used by Legendre. If $\tau$ denote the time occupied by a single oscillation, that is, the time of moving from $B$ to $A$, the maximum value of $\theta$, corresponding to $t = \frac{1}{2}\tau$, will be $\alpha$, the angle $ACO$ or $BCO$ in the diagram.

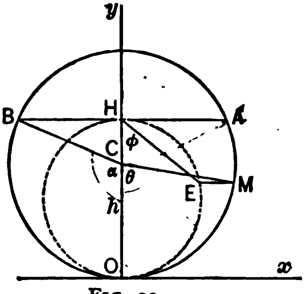

FIG. 99.

**378.** But, returning to equation (2), Art. 375, if we first eliminate $\theta$ instead of $y$, the use of another variable is suggested, which will reduce the expression for $t$ to a complete elliptic integral of the standard form.

From equation (3), we derive

$$\theta = 2 \sin^{-1} \frac{\sqrt{y}}{\sqrt{(2a)}},$$

whence

$$d\theta = \frac{dy}{\sqrt{y}\sqrt{(2a - y)}}.$$

Substitution in equation (2) gives

$$\frac{\sqrt{(2g)}}{a}dt = \frac{dy}{\sqrt{y}\sqrt{(2a-y)}\sqrt{(h-y)}}. \quad . \quad . \quad . \quad . \quad (6)$$

This would be reduced to the form given in Art. 375 by the substitution $y = 2a\sin^2\psi$. If however, instead of this, we make the similar substitution

$$y = h\sin^2\phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

so that

$$\sqrt{y} = \sqrt{h}.\sin\phi, \qquad \sqrt{(h-y)} = \sqrt{h}.\cos\phi,$$

and

$$dy = 2h\sin\phi\cos\phi\, d\phi,$$

equation (6) becomes

$$\frac{\sqrt{(2g)}}{a}dt = \frac{2d\phi}{\sqrt{(2a - h\sin^2\phi)}}.$$

Now putting

$$\frac{h}{2a} = k^2,$$

we find

$$\sqrt{\frac{g}{a}}\,t = \int_0 \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}, \quad . \quad . \quad . \quad (8)$$

in which the elliptic integral is of the standard form, since $k^2 < 1$ when $h < 2a$.

379. The auxiliary variable $\phi$, the amplitude of this elliptic integral, may be constructed, for a given position of $M$, by drawing the horizontal line $ME$, Fig. 99, to meet in $E$ the circumference of a circle described on $OH$ as a diameter, and then joining $EH$; for, denoting $OHE$ by $\phi$, the distince $OE$ is $h\sin\phi$, and $y = OE\sin\phi = h\sin^2\phi$, which is equation (7). While $t$ increases uniformly in equation (8), we must suppose $\phi$ to increase continuously, so that $E$ moves always in the same direction in the circle $OEH$, while $M$ travels back and forth over the arc $AOB$.

Equation (8) expresses $t$ as a function of $\phi$, and hence,

through equation (7), of $y$. Conversely, the expression of $y$ (and of other quantities belonging to a complete discussion of the motion), in terms of $t$, involves the functions inverse to the elliptic integrals, which are called *elliptic functions.*

## The Simple Pendulum.

380. The simple pendulum consists of a heavy particle made to describe a circle by means of a light rod or string, and oscillating generally over a small arc. If, as in Fig. 99, $\alpha$ is the extreme value of $\theta$, so that $2\alpha$ is the whole angle of swing, we have $h = 2a \sin^2 \frac{1}{2}\alpha$, since $OB = 2a \sin \frac{1}{2}\alpha$, and $OH = OB \sin \frac{1}{2}\alpha$; hence

$$k = \sqrt{\frac{h}{2a}} = \sin \tfrac{1}{2}\alpha.$$

Denoting the time of a single swing by $\tau$, the value of $t$ for the point $A$ is $\frac{1}{2}\tau$, and the corresponding value of $\phi$ is $\frac{1}{2}\pi$; hence, by equation (8), Art. 378,

$$\frac{1}{2}\sqrt{\frac{g}{a}}\,\tau = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = K, \quad . \quad . \quad . \quad (1)$$

$K$ denoting the complete elliptic integral regarded as a function of the modulus $k$.

To express $K$ in the form of a series, we have, by the Binomial Theorem,

$$(1 - k^2 \sin^2 \phi)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}k^2 \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4} k^4 \sin^4 \phi + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} k^6 \sin^6 \phi + \dots$$

Integrating each term in the definite integral by the formula

$$\int_0^{\frac{\pi}{2}} \sin^{2n} \phi \, d\phi = \frac{1 \cdot 3 \cdot 5 \dots (2n - 1)}{2 \cdot 4 \cdot 6 \dots 2n} \frac{\pi}{2}$$

(Int. Calc., Art. 86), we have

$$K = \frac{\pi}{2}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2}k^4 + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2}k^6 + \ldots\right].$$

Therefore, putting $l$ in place of $a$, equation (1) gives

$$\tau = \pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{4}\sin^2\tfrac{1}{2}\alpha + \frac{9}{64}\sin^4\tfrac{1}{2}\alpha + \ldots\right], \qquad (2)$$

where $l$ is the length of the pendulum, and $\frac{1}{2}\alpha$ the quarter angle of swing.

### The Seconds Pendulum.

**381.** If we put $\epsilon$ for the sum of all the terms but the first of the series, equation (2) becomes

$$\tau = \pi\sqrt{\frac{l}{g}}(1 + \epsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which, when $\epsilon = 0$, reduces to the expression for the cycloidal pendulum, the motion of which is shown in Art. 371 to be harmonic, and is said to be *isochronous* because the time is independent of the amplitude of swing. This is not true of the motion of the simple pendulum, but that motion is said to be *approximately isochronous* because $\epsilon$ involves, not the first power, but the square and higher powers of the small quantity $\sin\frac{1}{2}\alpha$.*

**382.** Putting $\tau = 1$, and $\epsilon = 0$, the first approximation to the length of the pendulum which beats seconds when $\alpha$ is small is

$$L = \frac{g}{\pi^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is usually given as *the length of the seconds pendulum.* It

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* As remarked in Art. 373, the tangential force, in the case of the cycloidal pendulum, is proportional to the arc, measured from the lowest point. In the present case, it is proportional to $\sin\theta$, and therefore very nearly proportional to the arc when $\theta$ is small.

is really the limiting length of the pendulum which beats seconds, when the arc of swing is indefinitely decreased.

Now, denoting by $l$ the length of the pendulum which beats seconds when swinging through the arc $2\alpha$, we find, by putting $\tau = 1$ in equation (3),

$$l = \frac{g}{\pi^2(1+\epsilon)^2} = L(1 - 2\epsilon + 3\epsilon^2 - \ldots),$$

in which (see Art. 381)

$$\epsilon = \frac{1}{4}\sin^2 \tfrac{1}{2}\alpha + \frac{9}{64}\sin^4 \tfrac{1}{2}\alpha + \ldots$$

Substituting, we have

$$l = L\left(1 - \frac{1}{2}\sin^2 \tfrac{1}{2}\alpha - \frac{3}{32}\sin^4 \tfrac{1}{2}\alpha - \ldots\right) \quad . \quad . \quad (5)$$

for the corrected length of the seconds pendulum designed to swing through the angle $2\alpha$.

**383.** By equation (3), the actual time of the beats of a "seconds pendulum," whose length is $L$, when swinging through the arc $2\alpha$, is $1 + \epsilon$; hence, if $N$ is the number of seconds in any definite interval, for example, in a day, the actual number of beats will be

$$n = \frac{N}{1+\epsilon} = N(1 - \epsilon + \epsilon^2 - \ldots).$$

We may therefore take

$$\epsilon N \qquad \text{or} \qquad \tfrac{1}{4}N\sin^2 \tfrac{1}{2}\alpha$$

approximately for the number of beats lost in $N$ seconds when the amplitude of swing is considerable. It follows that, if the length is already adjusted to the swing $2\alpha_0$, the number of beats lost when the swing is $2\alpha_1$ is

$$(\epsilon_1 - \epsilon_0)N \qquad \text{or} \qquad \tfrac{1}{4}N(\sin^2 \tfrac{1}{2}\alpha_1 - \sin^2 \tfrac{1}{2}\alpha_0),$$

which may also be written in either of the forms

$$\tfrac{1}{8}N(\cos\,\alpha_0 - \cos\,\alpha_1) \qquad \text{or} \qquad \tfrac{1}{4}N\sin\tfrac{1}{2}(\alpha_1 + \alpha_0)\sin\tfrac{1}{2}(\alpha_1 - \alpha_0)$$

### Comparison of Small Changes in $l$, $n$ and $g$.

384. If $n$ denotes the number of beats of a pendulum whose length is $l$ in $N$ seconds, we have $n\tau = N$, whence, from equation (2), Art. 381,

$$n = \frac{N}{\pi(1+\epsilon)}\sqrt{\frac{g}{l}}.$$

Supposing the angle of swing, and therefore $\epsilon$, to be unchanged, we have, by logarithmic differentiation, when $g$ and $l$ vary,

$$\frac{dn}{n} = \frac{dg}{2g} - \frac{dl}{2l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence, if $\varDelta l$ is a small error in the length of a pendulum intended to beat $n$ times in $N$ seconds, and $\varDelta n$ the consequent change in $n$ ($g$ remaining unchanged), we derive

$$\varDelta n = -\frac{n}{2}\,\frac{\varDelta l}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the negative sign showing that $n$ decreases as $l$ increases.

If $\varDelta n$ is known by observation, the error in $l$ is given by

$$\varDelta l = -\frac{2l\varDelta n}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For example, if the pendulum is intended to beat seconds, and $n$ is the number of seconds in a day, $\varDelta n$ is the number of beats gained in a day; then equation (3) gives the approximate error in $l$, which is too short if $\varDelta n$ is positive.

**385.** Again, for a small variation in $g$ while $l$ remains unaltered, equation (1) gives

$$\Delta g = \frac{2g\Delta n}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

a formula used in determining differences in the values of $g$ by comparing the number of beats in a given time of the same pendulum in different localities.

Experiments to determine $\Delta n$ to be used in this formula are called "pendulum experiments," and are usually made with a seconds pendulum. But this is not necessary, it is only essential that the length should be unaltered, and that $n$ and $n + \Delta n$ should be the number of beats made in precisely the same interval of time.

**386.** Since, as mentioned in Art. 352, the attraction of the earth upon a particle is inversely proportional to the square of the distance from the centre, denoting this distance by $r$, we have

$$g = g_0 \frac{R^2}{r^2},$$

where $g_0$ and $R$ are the values of $g$ and $r$ at the sea-level.

By logarithmic differentiation,

$$\frac{dg}{g} = -2\frac{dr}{r};$$

hence, if $\Delta g$ is the variation of gravity from $g_0$ for the height $\Delta r = h$ above sea-level, we have approximately

$$h = -\frac{R\Delta g}{2g_0};$$

but it is found that, when $h$ is the altitude of a place above sea-level, the local attraction of the mountain or table land modifies this result very considerably. Hence the result, namely,

$h = -\dfrac{R\Delta n}{n}$, of eliminating $\Delta g$ from this equation by means of equation (4) is not trustworthy as a means of determining geographical altitudes by pendulum experiments.

## EXAMPLES. XX.

1. A particle is allowed to slide from any point of a smooth hemisphere. Show that it will leave the hemisphere after describing one-third of its vertical height above the centre.

2. If the particle in the preceding example rests at the top of the hemisphere of radius $a$, what is the least horizontal velocity that must be given to it in order that it may leave the hemisphere at once? $\sqrt{(ag)}$.

3. If a particle starts from the cusp of a smooth inverted cycloid, prove that the pressure at any point is double what it would be if the particle started from that point.

4. Show that in the motion of the cycloidal pendulum the vertical velocity is greatest when one-half the vertical distance has been described.

5. A particle slides off a cycloid in erect position. Show that it will leave the curve when half the vertical height above the base has been described.

6. A heavy body is attached to a fixed point by means of a string 10 feet long. What is the least velocity it can have at its lowest point in order to describe a vertical circle keeping the string taut? $40^f/_s$.

7. A particle of weight $W$ attached to a fixed point by means of a string moves in a vertical circle. Determine the tension $P$ of the string at any point, using the notation of Art. 374.

$$\frac{P}{W} = \frac{2h}{a} - 2 + 3\cos\theta = \frac{2h + a - 3y}{a}.$$

8. Find the point at which the string becomes slack, and show that the result agrees with the construction given in Art. 370 for the centre of curvature of the parabola.

9. Show that the time of revolution of the conical pendulum is the same as that of a complete vibration through a small arc of the simple pendulum whose length is $h$.

10. Show that the motion of $E$ in Fig. 99 tends at the limit when $h$ is small to become uniform circular motion.

11. Prove that the time down the chord to the lowest point of a circle is to the time down the arc, when the arc is small, in the ratio $4 : \pi$.

12. A seconds pendulum in a railway car moving at the rate of 60 miles an hour on a circular track is observed to make 121 beats in two minutes. What is the radius of the circle? $\frac{1}{4}$ mile.

13. Find the length of the seconds pendulum, at a place where $g = 32.2$. 39.15 inches.

14. A clock which should beat seconds was found to lose 2 minutes a day at a place where $g = 32.2$. How many turns to the right should be given to a nut raising the pendulum-bob, the screw having 50 threads to the inch? 5.4375.

15. Taking the earth's radius to be 6366 kilometers, how many beats a day will a seconds pendulum lose at the top of the Eiffel Tower, which is 300 meters in height? 4.07.

16. A pendulum beats seconds when swinging through an angle of 6°. How many seconds will it lose a day when swinging through 8°, and through 10°? 11.56 sec.; 26.35 sec.

17. How much shorter than the "seconds pendulum" is that which beats seconds when swinging through an arc of 20°?

0.149 in.

# CHAPTER X.

## CENTRAL ORBITS.

### XXI.

### Free Motion about a Fixed Centre of Force.

**387.** We shall next consider the motion of a free particle subject to the action of a force directed always to or from a fixed point, or centre of force, and having an initial motion oblique to the direction of the force. As in § XVIII, we shall in this chapter suppose the intensity of the force to be a function solely of the distance of the particle from the centre of force. The line of the initial motion, together with the centre of force, determines a plane to which it is evident that the motion of the particle will be restricted, because the force has no component tending to move it out of the plane. The path of the particle is therefore a plane curve. It is called *the orbit* of the particle under the given force, and may or may not be a closed curve according to the law of the variation of the force.

**388.** In referring the particle to rectangular and to polar coordinates in this plane, we shall take the centre of force as the origin and pole, and the initial line coincident with the axis of $x$, so that we have the usual relations,

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Also, if $F$ is the force along $r$, since $\theta$ is its inclination to the axis of $x$, its components along the axes are

$$X = F\cos\theta, \qquad Y = F\sin\theta.$$

The acceleration $f$, or force acting upon a unit of mass, is by hypothesis a given function of $r$, the distance of the particle from the centre of force, and may be written $f(r)$. If $m$ is the mass of the particle, the force is

$$F = mf(r);$$

hence we have

$$\frac{X}{m} = \frac{F\cos\theta}{m} = f(r)\frac{x}{r},$$

$$\frac{Y}{m} = \frac{F\sin\theta}{m} = f(r)\frac{y}{r};$$

and the equations of the two components of the motion are

$$\frac{d^2x}{dt^2} = f(r)\frac{x}{r}, \qquad \frac{d^2y}{dt^2} = f(r)\frac{y}{r}.$$

## Attraction Directly Proportional to the Distance.

389. We shall first consider the case of an attractive force whose intensity is proportional to the distance from the centre, and which, as we have seen in Art. 337, produces harmonic motion when the initial velocity has no component transverse to its direction. Putting $f(r) = -\mu r$, the two equations of motion become

$$\frac{d^2x}{dt^2} = -\mu x, \qquad \frac{d^2y}{dt^2} = -\mu y,$$

each of which is of the same form as that of Art. 335. Hence putting $\mu = n^2$ as in that article, and denoting by $a$ the value of $x$ which corresponds to the component velocity $\frac{dx}{dt} = 0$ (that is, the maximum value of $x$), we have

$$x = a\sin(nt + C).$$

Similarly, if $b$ is the maximum value of $y$, we have

$$y = b \sin (nt + C').$$

If we take for the origin of time the instant when $x = a$, corresponding to $s = a$ in Art. 336, we have $C = \frac{1}{2}\pi$; and, denoting the corresponding value of $C'$ by $\alpha$, the equations become

$$x = a \cos nt, \qquad (1)$$

$$y = b \sin (nt + \alpha). \qquad (2)$$

**390.** It follows that the motion of the particle is a combination of two harmonic motions having the same period, namely,

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{\mu}}.$$

See Art. 338. To values of $t$ differing by any multiple of this period correspond the same values of $x$ and the same values of $y$; hence the particle returns to the same position periodically; therefore its path or orbit is a closed curve, which, as represented in Fig. 100, is inscribed in the rectangle whose sides are $2a$ and $2b$, parallel to the axes, and whose centre is the origin.

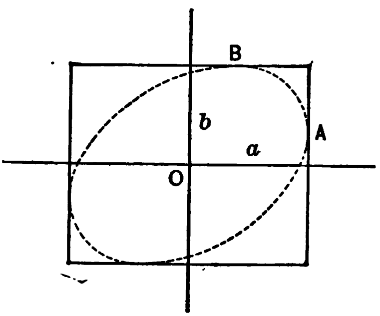


FIG. 100.

Elimination of $t$ between equations (1) and (2) obviously gives an equation of the second degree between $x$ and $y$, hence the orbit is an ellipse. The amplitudes $a$ and $b$ determine the size of the circumscribing rectangle, and the constant $\alpha$, depending on the difference of the phases of the harmonic motions, determines the positions of $A$ and $B$, the points of contact with its sides.

391. In particular, if $\alpha = 0$, so that the phases differ by a quarter period, the equations become

$$\left.\begin{array}{l} x = a\cos nt, \\ y = b\sin nt. \end{array}\right\} \qquad (1)$$

Eliminating $t$, we find the orbit, in this case, to be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (2)$$

of which the semi-axes are $a$ and $b$, and this ellipse is described by the particle in such a manner that the eccentric angle, which is $nt$ in equations (1), increases uniformly at the rate $n$, completing a revolution in the time $\frac{2\pi}{n}$.

Since in any case the orbit is an ellipse and the direction of the coordinate axes is arbitrary, we shall always obtain this result if the axes are taken in the direction of the axes of the ellipse. Therefore the motion is always of the character described above; namely, *elliptical with uniform increase of the eccentric angle.* It follows also that the resolved part of such a motion in any direction is harmonic.

392. If in the equations of the preceding article $b = a$, the orbit becomes the circle $x^2 + y^2 = a^2$ described with uniform motion, the motion being in fact the same as that of the auxiliary point $P$ in Fig. 91, p. 268. In this case, since $r$ is constantly equal to $a$, the acceleration toward the centre takes the constant value $f = \mu a = n^2 a$, and the constant linear velocity is $V = na$; hence, in this case, we have

$$f = \frac{v^2}{a}.$$

This is the value of the centripetal force in uniform circular motion, which, as mentioned in Art. 358, may be regarded as producing the two rectangular components of the motion.

In the motion of the conical pendulum the resolved part of the motion in any given direction is strictly harmonic and has the same period for all directions, which, as we have seen in Ex. XX. 9, is the approximate time, for small amplitudes, of the complete vibration of a simple pendulum of length $h$. When the amplitude is not the same in different directions, the motion is, if the variation of $h$ be neglected, the elliptical harmonic motion described in the preceding articles. Thus, when a plummet suspended by a string is drawn aside and let go with a slight lateral motion, it will describe a curve approximating to an ellipse.*

## Acceleration Along and Perpendicular to the Radius Vector.

393. In the general treatment of motion under a central force we have special occasion to employ the expressions for the acceleration, in the direction of, and perpendicular to, the radius vector.

Acceleration is defined in Art. 39 as the rate of change in the velocity, direction as well as amount being considered. Hence, when the velocity is resolved into rectangular components, it cannot be inferred that the derivatives or rates of the components give the component accelerations in the given directions, *unless these directions are constant.* Thus, the accelerations in the fixed directions of the axes are (see Art. 42) simply $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, the derivatives with respect to $t$ of the corresponding velocities; but that along the radius vector, of which the direction is variable, is *not* simply $\frac{dr^2}{dt^2}$. In fact, we have already seen that the latter acceleration does not vanish when $r$ is constant, but becomes the centripetal acceleration derived in Art. 40.

* The slight departure of the force from strict proportionality to the distance causes the axes of the apparent ellipse to rotate slowly in the direction of the motion.

394. Since the projection in any direction of a vector is the sum of the projections in that direction of its component vectors, the acceleration in the direction of $r$, which makes the angle $\theta$ with the axis of $x$, is the sum of the resolved parts in that direction of the component accelerations along the axes of $x$ and $y$. Hence it is

$$\frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta.$$

In like manner, the acceleration at right angles to the radius vector, that is, in the direction $\theta + 90°$ (which we take as the position direction for the transverse acceleration because it is the direction of $\theta$ increasing), is

$$-\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta.$$

To express these accelerations in polar coordinates, it is necessary first to obtain the polar expressions for $\dfrac{d^2x}{dt^2}$ and $\dfrac{d^2y}{dt^2}$. From

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta,$$

we have, by differentiation,

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt},$$

$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt};$$

whence

$$\frac{d^2x}{dt^2} = \frac{d^2r}{dt^2}\cos\theta - 2\frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2},$$

$$\frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}\sin\theta + 2\frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}.$$

Substituting these values in the expressions for the two accelerations, we find for that in the direction of the radius vector

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(which when $r$ is constant reduces to the expression for centripetal acceleration), and for that transverse to $r$

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2},$$

which may also be written in the form

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

## Area described by the Radius Vector under the Action of a Central Force.

395. Denoting the components of the force $F$ acting upon a particle of mass $m$ along, and perpendicular to, the radius vector by $P$ and $Q$ respectively, we may take for the two general equations of motion

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{P}{m},$$

and

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = \frac{Q}{m}.$$

In the case of a central force, $P$ becomes $F = mf(r)$ (see Art. 388) and $Q = 0$; so that the second equation becomes

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0.$$

This is directly integrable, giving

$$r^2 \frac{d\theta}{dt} = h,$$

in which the constant of integration $h$ is arbitrary.

Now, since $r^2 d\theta$ is double the differential of the area swept over by the radius vector, this equation shows that, in the case of a central force, the rate at which area is described by the radius vector is constant. In other words, *the area described by the radius vector is proportional to the time*, and the constant $h$ is *double the area described in a unit of time.*

396. The first integral found above is also readily obtained in rectangular coordinates. The rectangular equations of motion for a central force, Art. 388, are

$$\frac{d^2x}{dt^2} = f(r)\frac{x}{r}, \qquad \frac{d^2y}{dt^2} = f(r)\frac{y}{r}.$$

Eliminating $f(r)$, we derive

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which is an exact differential equation, giving the first integral

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The constant $h$ in this equation has the same meaning as in the preceding article; for the equation may be written

$$\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)\frac{ds}{dt} = h,$$

in which the quantity enclosed in parentheses is the expression for the perpendicular $p$ from the origin upon the tangent (Diff. Calc., Art. 316), so that the equation is

$$pv = h. \qquad \qquad (3)$$

Now if we join the extremities of the line representing the velocity to the origin, $pv$ is double the area of the triangle thus formed; hence $h$ is double the area which would be described by the radius vector in a unit of time if the velocity were uniform; that is to say, double the rate at which area is being swept over by the radius vector.

397. This important theorem was proved geometrically by Newton substantially as follows: Let $AB$, Fig. 101, be the line which a moving particle describes by virtue of its velocity in a certain interval of time. Then, if there were no force acting, it would in the next equal interval of time describe the equal line $Bc$ in $AB$ produced, and the areas swept over by the line joining the particle to any fixed point $S$ would be the equal triangles $ASB$ and $BSc$. Now, when the body is at $B$, let a force act, with a single impulse toward $S$, giving the body a velocity which would cause it to move over the space $BD$ in the interval of time considered. Then, by the second law of motion, the body will actually describe the line $BC$, the diagonal of the parallelogram $BDCc$, and the radius vector connecting the body with $S$ will describe the triangle $BSC$. But since $cC$ is parallel to $BS$, the area of this triangle is equal to that of $BSc$, and therefore to that of $ASB$. Hence a force suddenly acting toward $S$ does not affect the rate at which area is generated by the radius vector, and it is readily inferred that a force acting continuously toward $S$ will not affect this rate.

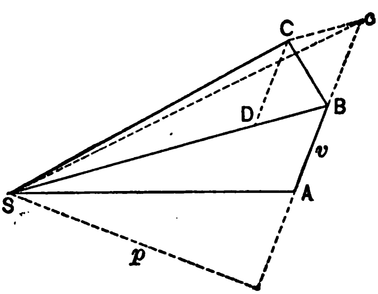


FIG. 101.

Conversely, Newton proved that, if equal areas are described in equal times by the radius vector joining the body to a fixed point, the force acting upon it must always be directed to that point. Therefore, from Kepler's law that the radii vectores of the planets in their orbits around the sun describe equal areas in equal times, he inferred that the force acting on them is always directed to the sun.

## The Differential Polar Equation of the Orbit.

398. The general equations of motion in a plane imply four constants of integration, but by means of the integration effected in Art. 395, the system of equations in polar coordinates for the case of a central force has become

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = f(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$r^2\frac{d\theta}{dt} = h, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which one constant $h$ has been introduced, and three are implied. Since $t$ does not occur explicitly in these equations, it is possible by means of equation (2) to eliminate $t$ from equation (1), and thus obtain the differential equation of the orbit, or relation between $r$ and $\theta$. In doing this, $\theta$ will as usual be taken as the independent variable, so that the process is that of changing the independent variable in equation (1) from $t$ to $\theta$.

399. The result is found to take a simpler form if, at the same time, we change the dependent variable from $r$ to its reciprocal, which we shall denote by $u$. Then $f(r)$ becomes a given function of $u$, and we have

$$r = \frac{1}{u}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$f(r) = f\left(\frac{1}{u}\right) = \phi(u). \quad \ldots \ldots \quad (4)$$

Equation (2) now becomes

$$\frac{d\theta}{dt} = hu^2.$$

Differentiating equation (3), we have, by virtue of this,

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -h\frac{du}{d\theta}.$$

Again, differentiating this last equation,

$$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting in equation (1), we have

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = \phi(u),$$

or

$$\frac{d^2u}{d\theta^2} + u = -\frac{\phi(u)}{h^2u^2}, \quad \ldots \ldots \quad (5)$$

which is the differential equation required. The integration of this equation, when the function $\phi$ is known, introduces two more constants; thus, three constants occur in the equation of the orbit, their values depending upon the initial circumstances of motion—for example, upon the distance, direction and velocity of the particle when it crosses the initial line.

Finally, the integration of equation (2), after $r$ has been expressed in terms of $\theta$, introduces the fourth constant.

## The Central Force under which a given Orbit is described.

**400.** The equation found above enables us to determine the law of variation with the distance in accordance with which a given orbit may be described about a given point as the centre of force. The equation of the orbit is supposed given in polar coordinates, the centre of force being the pole. Since the case of an attractive force is the more usual one, we shall put $P$ for the *attraction* acting on a unit of mass, thus:

$$P = -f(r) = -\phi(u).$$

Then equation (5) of the preceding article may be written

$$P = h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right],$$

in which $u$, the reciprocal of $r$, is given as a function of $\theta$ by the equation of the orbit, and $h^2$ is an arbitrary positive constant. The form assumed by the result is therefore that $P$ is proportional to a certain function of $r$.

The value given to the arbitrary constant determines the constant rate at which area is swept over by the radius vector. and thus determines the velocity at every point of the orbit.

**401.** For example, let us find the attractive force under which a body can describe an ellipse of which the centre of force is a focus. The equation of the ellipse, when the pole is a focus is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}, \quad \text{or} \quad u = \frac{1+e\cos\theta}{a(1-e^2)}; \quad . \quad . \quad (1)$$

whence we find

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{a(1-e^2)}.$$

Therefore, in this case, $P = \dfrac{h^2u^2}{a(1-e^2)}$, or, putting

$$\mu = \frac{h^2}{a(1-e^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$P = \frac{\mu}{r^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is to say, *the force varies inversely as the square of the distance.* The arbitrary constant $\mu$ in equation (3) represents *the intensity* of the attraction, that is the force acting on a unit of mass at a unit's distance. Its value determines the velocity for, by equation (2), $h^2 = a\mu(1 - e^2)$, and since $h = pv$,

$$v = \frac{\sqrt{\{a\mu(1 - e^2)\}}}{p}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $p$ is the perpendicular upon the tangent. For instance, at the nearest vertex where $p = r = a(1 - e)$, we find

$$v = \sqrt{\frac{\mu(1 + e)}{a(1 - e)}}.$$

## The Equation of Energy.

**402.** When a body moving in a plane is referred to *rectangular* axes, if we denote the component velocities in the directions of the axes by

$$v_x = \frac{dx}{dt}, \qquad v_y = \frac{dy}{dt},$$

we have

$$v^2 = v_x^2 + v_y^2.$$

Whence, as in Art. 323,

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_x^2 + \tfrac{1}{2}mv_y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the quantities in the second member may be called the component or *resolved kinetic energies*, since each of them is the energy which the body would have if it were moving with one of the rectangular components, to which we have restricted the term *resolved* velocities.

Now, the general equations of motion for any value of $F$ may be written

$$m\frac{dv_x}{dt} = X, \qquad m\frac{dv_y}{dt} = Y.$$

Treating each of these as in Art. 334, we have

$$mv_x\frac{dv_x}{dt} = X\frac{dx}{dt}, \qquad mv_y\frac{dv_y}{dt} = Y\frac{dy}{dt}.$$

whence

$$d(\tfrac{1}{2}mv_x^2) = Xdx, \qquad d(\tfrac{1}{2}mv_y^2) = Ydy. \quad . \quad . \quad . \quad (2)$$

These equations separately express that the work-rate of each component force is the same as the rate of increase of the corresponding component of kinetic energy, and their sum,

$$d(\tfrac{1}{2}mv^2) = Xdx + Ydy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in like manner shows that the rate of the whole kinetic energy is equal to the actual work-rate of the whole force. (Compare Art. 275.)

**403.** In the case of a central force, we have, as in Art. 388,

$$X = mf(r)\frac{x}{r}, \qquad Y = mf(r)\frac{y}{r},$$

and equation (3) above becomes

$$d(\tfrac{1}{2}mv^2) = m\frac{f(r)}{r}(xdx + ydy).$$

But, since $x^2 + y^2 = r^2$, $xdx + ydy = rdr$, hence

$$d(\tfrac{1}{2}mv^2) = mf(r)dr.$$

This equation is integrable, because the second member contains the single variable $r$. It is in fact $Fdr$, the element of work done, and its integral is the work function $V$ of Art. 278. Hence if $v_1$ and $v_2$ are the velocities with which the body passes any two points of its orbit, and $r_1$, $r_2$ the distances of the points from the centre of force, we have by integration

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = \int_{r_1}^{r_2} Fdr = V_2 - V_1. \quad . \quad . \quad . \quad . \quad (1)$$

It follows that the gain or loss of kinetic energy in passing over an arc of the orbit depends only upon the distances of the extremities from the centre. In other words, given the initial velocity and distance, the kinetic energy, and therefore *the velocity, at any point of the orbit, depends only upon its distance from the centre of force.*

404. Let

$$U = \int_a Pdr\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

which, for the attractive force $P = -F$, is the function of $r$, the distance from the centre of force, which was defined in Art. 279 as *the potential function.* The value of the integral taken between limits is the difference between the potential energies corresponding to the limits, and therefore $U$, as written above, is the potential energy at the distance $r$, when so taken as to vanish at the distance $a$. The integral of equation (1) may now be written in the form

$$\tfrac{1}{2}mv^2 + U = C, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

which expresses that *the sum of the kinetic and potential energies of the body is constant.* This is the equation of energy, and shows that the Principle of Conservation of Energy in its mechanical forms, which was proved in Art. 294, for a body moving in a straight line passing through the centre of force, *extends also to the case of a body describing an orbit under the action of a central force.*

### The Circle of Total Energy, or of Zero Velocity.

405. Since $U$ is a function of $r$, the points for which it has a given value lie on the circumference of a circle whose centre is at the centre of force. In other words, the *equipotential lines*

are the circumferences of concentric circles. For an attractive force, the potential increases outward from the centre of force. Suppose, in the first place, that there exists an equipotential circle on which the value of the potential is equal to that of the constant total energy of the body in its orbit, represented by $C$ in equation (3) of the preceding article. This may be called *the circle of total energy.* If a body of the same mass as that describing the orbit fall freely from rest on the circumference of this circle, its constant total energy will be the same as that of the body in the orbit. When the two bodies are at the same distance from the centre, so that they have the same potential energy, they will also have the same kinetic energy, and therefore the same velocity. Thus, at every point of the orbit, the body has the velocity which would be acquired by falling freely under the action of the given force from the circle of total energy to its actual position. This velocity may be called *the velocity due* to the given circle.

**406.** If a body having the velocity due to a certain equipotential circle were constrained to move in a smooth path, the forces of constraint would do no work; hence, the total energy would remain fixed, and the velocity at any given distance from the centre would still be that due to the given equipotential circle, or we may say to the given *level.* Wherever the path reached the circle of total energy its kinetic energy and, therefore, its velocity would vanish. Gompare Art. 369. This circle is, therefore, also sometimes called *the circle of zero velocity.*

**407.** A body moving in a free orbit, however, can never reach the circle of total energy; for the relation $pv = h$ shows that $v$ can never vanish at a finite distance. Hence, when the circle of total energy exists, the orbit is entirely enclosed within it.

For example, in the case of the attraction directly proportional to the distance, or $P = \mu r$, the potential is, as in Art. 281, $U = \frac{1}{2}\mu r^2$ (taking $m = 1$), which increases without limit as $r$ increases. The circle of total energy, therefore, always exists in this case. For motion in a line with the centre of force, which is simple harmonic motion, it is the circle whose radius is $a$, the amplitude, so that the body just reaches it; but for the body de-

scribing an orbit, as in Art. 390 or in Art. 391, it is the circle whose radius is $\sqrt{(a^2 + b^2)}$, which encloses without touching the elliptical orbit.

**408.** When the law of variation of the force is such that the potential at infinity is finite, (of which the gravitation potential, Art. 350, affords an example,) the total energy in the orbit may be equal to, or it may exceed, the potential at infinity. In the first of these cases, the circle of total energy or of zero velocity is at an infinite distance, and the velocity at every point of the orbit is that *due to infinity*.

A body projected with such a velocity directly away from the centre of force would never cease to recede from it, and it is possible, although not necessarily the case, that a body describing an orbit with the velocity due to infinity may also so recede, the velocity in that case approaching zero as a limit. The relation $pv = h$ shows that, under these circumstances, $p$ increases without limit; that is, the infinite branch of the orbit is of parabolic character.

**409.** So too, when the total energy exceeds the potential at infinity, or, what is the same thing, when the velocity in the orbit exceeds that due to infinity, the body may, but does not necessarily, recede without limit. If it does so recede, its kinetic energy, and therefore its velocity, will approach a finite limit; and the relation $pv = h$ shows that the perpendicular from the centre of force upon the tangent will also approach a finite limit; that is to say, the orbit will have an asymptote whose distance from the centre is the limiting value of $p$.

## The First Integral of the Equation of the Orbit.

**410.** We have seen in Art. 400 that, supposing $m = 1$, the differential equation of the orbit may be written in the form

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The first integral is found, as in similar cases, by direct integration after multiplying by $2\dfrac{du}{d\theta}$; thus we have

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\int\frac{Pdu}{u^2}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, since $r = \dfrac{1}{u}$, $dr = -\dfrac{du}{u^2}$, whence

$$\int\frac{Pdu}{u^2} = -\int Pdr = -U + C$$

(Art. 404), and therefore equation (2) may be written

$$\frac{h^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U = C. \quad . \quad . \quad . \quad . \quad (3)$$

The form of this equation shows that it is identical with equation (3) of Art. 404, hence the first term is an expression for $\frac{1}{2}v^2$,* the kinetic energy when $m = 1$. The constant $C$ is therefore the total energy of a unit mass in the orbit.

**411.** If the radius of the circle of zero velocity be given, the constant implied in equation (2) my be determined by simply using the corresponding value of $u$ as the lower limit in the indefinite integral. For this makes the integral the expression for the potential, *so taken as to vanish on the given circle*; and, when this is done, $C$ vanishes for the given orbit. Thus, for example, *the equation of the orbit in which the velocity is that due to infinity is*

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2}{h^2}\int_0\frac{Pdu}{u^2}.$$

---

* Accordingly, $v^2 = \dfrac{h^2}{p^2}$ and $\dfrac{1}{p^2} = \left(\dfrac{du}{d\theta}\right)^2 + u^2$, as shown in Diff. Calc., Art. 321.

## The Apsides of the Orbit.

**412.** A point at which the radius vector is normal to the orbit is called an *apse*. The corresponding value of $r$ is called an *apsidal distance*, and is either a maximum or a minimum value. When $r$ is a maximum, $u$ is a minimum, and *vice versa*; hence the apsidal distances may be found by putting $\frac{du}{d\theta} = 0$ in equation (3), Art. 410.

That equation may be written in the form

$$\left[\frac{du}{d\theta}\right]^2 = \psi(u), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where

$$\psi(u) = \frac{2}{h^2}(C - U) - u^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Since we are concerned only with positive* values of $u$, it follows that an apsidal value $u_0$ is a positive root of the equation

$$\psi(u_0) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Again, equation (1) shows that there can be no orbit having values of $C$ and $h$ which make $\psi(u)$ negative for *all* positive values of $u$. If $\psi(u)$ is positive for all such values, the orbit will in one direction recede to infinity, and in the other direction pass to the centre of force. But, when there is an apsidal value $u_0$, $\psi(u)$ will

* Unless $P$ is an odd function of $r$, so as to change its sign, but not its numerical value, when $r$ is changed to $-r$, negative values correspond to a different law of force. For example, if $P$ is an even function, negative values of $r$ imply a repulsive force with the same law of variation. Compare the laws of force treated in Art. 335 and in Art 346; in the former case $s$ can change sign, in the latter it cannot.

generally change sign as $u$ passes through this value; hence the circle whose radius is the apsidal distance is the boundary of a region which the orbit cannot enter. The orbit will now pass in both directions to infinity, or to the centre of force as the case may be, unless it reaches another circle on which $\psi(u)$ vanishes. In this last case, the orbit is confined to the annular space between these two circles, and their radii are the maximum and minimum values of $r$. Thus, *there cannot be more than two apsidal distances* in a given orbit, although there may be any number of apsides.

**413.** If $\beta$ is the value of $\theta$ for an apsidal value $u_0$, it is obvious that, for neighboring values of $u$, on that side for which $\psi(u)$ is positive, there are two real values of $\theta$ which become equal when $u = u_0$. Now, by equation (1), we have

$$d\theta = \pm \frac{du}{\sqrt{\psi(u)}}.$$

Integrating, we have for the equation of the orbit

$$\theta = \beta \pm \int_{u_0} \frac{du}{\sqrt{\psi(u)}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This equation expresses the two values of $\theta$, which become equal* when $u = u_0$. Thus, the third constant of integration, $\beta$, now introduced into the equation of the orbit, determines simply the direction of an apsidal radius vector, and has no connection with the shape of the orbit, which depends solely upon the constants $h$ and $C$.

* If $u_0$ in the integral were not an apsidal value of $u$, the values of the constant to be used with the upper and lower sign, in a given orbit, would be different. Owing, however, to the multiple values of the integral, the orbit is completely represented when a single sign is employed, whether the lower limit is an apsidal value or not.

414. *A central orbit is symmetrical to every apsidal radius vector.* For, taking $\beta = 0$, that is to say, reckoning $\theta$ from the direction of the apsidal radius vector, equation (4) shows that to every point $(u, \theta)$ on the orbit there corresponds a point $(u, -\theta)$ also on the orbit, but this is the point symmetrically situated to $(u, \theta)$.

The form of equation (2), Art. 398, shows that the fourth constant of integration, referred to in Art. 399, determines simply *the epoch* or time of passing a given point of the orbit; and, if we reckon the time from the instant when the body passes the apse, the times $t$ and $-t$ correspond to the symmetrically situated points. It is obvious, also, that the orbit may be described in either direction, so that the epoch should include the direction of motion as well as the time of passing the given point.

415. In the case of an orbit having two apsidal distances, let $u_1$ be the other apsidal value of $u$, then, by equation (4), the angle between two consecutive apsidal radii is

$$\int_{u_0}^{u_1} \frac{du}{\sqrt{\psi(u)}}.$$

This is called *the apsidal angle.* As we have already seen, it is necessary not only that $u_0$ and $u_1$ should be roots of the equation $\psi(u) = 0$, but that the value of the functions should be *positive* for intermediate values of $u$.

## The Radius of Curvature at an Apse.

416. At an apse the radius vector coincides with the perpendicular upon the tangent, that is $p = r$, and the centrifugal force is directly opposed to, and therefore in equilibrium with, the attractive force $P$. Let $v_0$ be the velocity, and $\rho_0$ the radius of curvature at the apse whose distance is $r_0$, $P_0$ being the corresponding value of $P$; then, from $h = pv$, we have

$$v_0 = \frac{h}{r_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, from the expression for centrifugal force,

$$\frac{v_0^2}{\rho_0} = P_0 ;$$

whence

$$\rho_0 = \frac{h^2}{r_0^2 P_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If the value of $\rho_0$ thus found *exceeds* $r_0$, the centre of curvature lies beyond the centre of force, and the orbit lies outside of the circle whose radius is $r^0$; that is, $r_0$ is a *minimum* apsidal distance. This corresponds to the case in which $\psi(u)$ has positive values for greater values of $r$, that is, for values of $u$ *less* than $u_0$. On the other hand, if $\rho_0$ is less than $r_0$, the latter is a *maximum* apsidal distance.

417. If a second apsidal distance $r_1$ be possible, but none whose value lies between $r_1$ and $r_0$, $r_1$ will be found to be a maximum or a minimum, according as $r_0$ is a minimum or a maximum. If the maximum be greater than the minimum, $\psi(u)$ will be positive in the annular space between the apsidal circles, and the case is that of an orbit with two apsidal values. If the contrary be the case, $\psi(u)$ will be negative in the annular space and we infer that, with the same values of $h$ and $C$ (namely, those employed in forming the function $\psi$), two orbits are possible—one situated beyond the annular space between the apsidal circles and passing to infinity or to another apsidal distance, the other within the smaller circle, and passing to the centre of force or to another apsidal distance.

## Circular Orbits.

418. For any central attraction depending solely upon the distance, a circular orbit with a given radius is possible if the velocity be properly determined. For this purpose it is only necessary to equate the centrifugal force to the attraction at the given distance. Denoting the required velocity by $V$, we have

$$\frac{V^2}{r} = P; \qquad \text{whence} \qquad V = \sqrt{(rP)}$$

is *the circular velocity* at the distance $r$.

**419.** When a circular orbit is regarded as a special case of the orbit described under the given law of force, the given value of $r$ corresponds to the apsidal distance $r_0$ of Art. 416, and $p_0 = r_0$. We must therefore suppose $C$ and $h$ to have been so taken that, from equation (2), Art. 416 (or from $h = Vp = Vr$),

$$h^2 = r_0^3 P_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while, at the same time, the reciprocal of the given value of $r_0$ is a root of the equation $\psi(u) = 0$. The value of $h$ is therefore determined by the equation just written, and then substituting in the equation $\psi(u_0) = 0$, we have (see Art. 412)

$$C = U_0 + \tfrac{1}{2} r_0 P_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

With these values of $C$ and $h$, $u_0$ becomes a double root of $\psi(u) = 0$, and the function does not change sign as $u$ passes through the value $u_0$. If its value is negative for values of $u$ on each side of $u_0$, the orbit is the limiting form of orbits having two nearly equal apsidal distances, and lying in an annular space where $\psi(u)$ is positive. These orbits, as the space narrows, approximate more and more nearly to the circle. The circle is in this case said to be described with *kinetic stability*. In the opposite case, that is, when the values of $\psi(u)$ are positive for values of $u$ on each side of $u_0$, we have the limiting form of the second case mentioned in Art. 417, in which the annular space (which vanishes at the limit) is one in which $\psi(u)$ is negative, so that orbits approximating to a circle are not possible. The circular orbit is in this case described with *kinetic instability;* that is to say, the slightest change in the direction or velocity of the body will cause the orbit to assume a totally different form; namely, one which has *only one* apsidal distance equal to $r_0$, so that (unless $\psi(u) = 0$ has other roots besides those which have become equal) it will become one which passes either to infinity or to the centre.

## Attraction Inversely Proportional to the Square of the Distance.

420. The most important case of central orbits is that in which the force is an attraction varying inversely as the square of the distance, which is the actual law of gravity. Putting, in equation (1), Art. 410,

$$P = \frac{\mu}{r^2} = \mu u^2$$

(so that $\mu$ is the attraction acting upon the unit mass at the unit of distance), the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}, \quad \dots \dots \dots \quad (1)$$

or, putting

$$u' = u - \frac{\mu}{h^2},$$

$$\frac{d^2u'}{d\theta^2} + u' = 0.$$

Multiplying by $2\dfrac{du'}{d\theta}$, and integrating, we may write

$$\left(\frac{du'}{d\theta}\right)^2 + u'^2 = c^2, \quad \dots \dots \dots \quad (2)$$

since the constant of integration is necessarily positive. We have, then,

$$d\theta = \frac{du'}{\sqrt{(c^2 - u'^2)}},$$

and, integrating again,

$$\theta + \beta = \sin^{-1}\frac{u'}{c},$$

or

$$u - \frac{\mu}{h^2} = c \sin(\theta + \beta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the equation of the orbit involving three arbitrary constants $h$, $c$ and $\beta$. A maximum value of $u$, and hence a minimum value of $r$, occurs when $\theta + \beta = \frac{1}{2}\pi$. Therefore, if we take the prime vector (corresponding to $\theta = 0$) in the direction of such an apsidal value of $r$, we shall have $\beta = \frac{1}{2}\pi$, and equation (3) may be written in the form

$$u = \frac{\mu}{h^2}(1 + e \cos\theta), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which the constant $e$ replaces the positive quantity $\frac{ch^2}{\mu}$. This is equivalent to

$$r = \frac{\frac{h^2}{\mu}}{1 + e \cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of a conic, referred to a focus. The orbit is, therefore, a conic whose eccentricity is $e$ and semi-latus rectum $\frac{h^2}{\mu}$, the centre of force being at a focus.

**421.** The potential function for this force is

$$U = \int_\infty \frac{\mu}{r^2} dr = -\frac{\mu}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

taken as in Art. 350, so as to vanish when $r$ is infinite. The equation of energy, Art. 404, taking $m = 1$, is therefore

$$\tfrac{1}{2}v^2 - \frac{\mu}{r} = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

To express the constant $C$ in terms of those already introduced, we notice that the value of the apsidal distance, or minimum value of $r$, corresponding to $\theta = 0$ in equation (5), is

$$r_0 = \frac{h^2}{\mu(1 + e)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and the velocity at this apse is

$$v_0 = \frac{h}{r_0} = \frac{\mu\, 1 + e)}{h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Substituting these values of $r$ and $v$ in equation (7), we find

$$C = \frac{\mu^2(1 + e)^2}{2h^2} - \frac{\mu^2(1 + e)}{h^2} = \frac{\mu^2(e^2 - 1)}{2h^2}, \quad . \quad . \quad (10)$$

and introducing this value in equation (7), we have

$$v^2 = \frac{2\mu}{r} - \frac{\mu^2(1 - e^2)}{h^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

which determines the velocity at any point of the orbit.

**422.** The orbit is an ellipse, a parabola or an hyperbola, according as the eccentricity $e$ is less than, equal to or greater than unity; that is (see equation (10) ), according as the total energy is less than, equal to or greater than the potential at infinity. Putting $e = 1$ in equation (11), we have for the velocity in a parabolic orbit

$$v' = \sqrt{\frac{2\mu}{r}},$$

which in fact is the velocity from infinity corresponding to the distance $r$. The criterion with respect to the nature of the orbit may therefore be also stated as follows: The orbit described by a body projected from any point will be an ellipse, a parabola or an hyperbola, according as the velocity is less than, equal to

or greater than the velocity $v'$ which would be acquired in falling from rest at infinity to the point of projection.

## Elliptical Motion.

**423.** Let us now suppose the orbit to be an ellipse, as in the case of a planet revolving about the sun, which is the centre of force, situated at one of the foci. Let Fig. 102 represent such an orbit. The major axis is the line of apsides, and its extremities $A$ and $B$, which are the points of the orbit nearest to and farthest from the sun at the focus $S$, are called respectively *the perihelion* and *the aphelion*. Their distances are the values of $r$ in the equation of the orbit

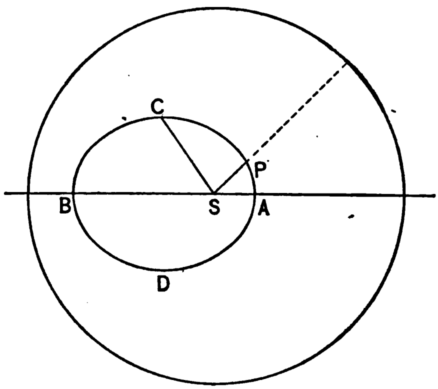


FIG. 102.

$$r = \frac{h^2}{\mu(1 + e\cos\theta)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

corresponding to $\theta = 0$ and $\theta = 180°$ (which is the apsidal angle), namely,

$$SA = \frac{h^2}{\mu(1 + e)} \quad \text{and} \quad SB = \frac{h^2}{\mu(1 - e)}. \quad . \quad (2)$$

Denoting, as usual, the major semi-axis by $a$, we have

$$a = \tfrac{1}{2}(SA + SB) = \frac{h^2}{\mu(1 - e^2)}. \quad . \quad . \quad . \quad (3)$$

**424.** By equation (6), Art. 421, the potential energy at the distance $2a$ from $S$ is

$$-\frac{\mu}{2a} = -\frac{\mu^2(1-e^2)}{2h^2},$$

which, by equation (10), is the value of $C$, the total energy. It follows that the major axis $2a$ is the radius of the circle of total energy, and that equation (11), Art. 421, for determining the velocity at a given distance, may, for the ellipse, be written in the form

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The circle of total energy or zero velocity is drawn in the diagram; the velocity at $P$ is that which would be acquired by falling from rest through the dotted line. The major semi-axis $a$, which is an arithmetical mean between the perihelion and aphelion distances is called *the mean distance* of the planet.

Equation (4) shows that the orbits described by all bodies projected from the same point with the same velocity will have the same mean distance. The circle of zero velocity will be the same for all of these orbits, just as the directrix, which plays the same part in the case of a constant force (see Art. 322), is the same for the trajectories of all projectiles having the same initial velocity and point of projection.

**425.** When the planet is at its mean distance, that is, when $r = a$, equation (4) of the preceding article gives

$$v^2 = \frac{2\mu}{a} - \frac{\mu}{a} = \frac{\mu}{a}.$$

This equation gives also the constant velocity in a circular

orbit whose radius is $a$, agreeing with Art. 418, when we put $P = \frac{\mu}{r^2}$. Hence, *the velocity at the mean distance is the same as the circular velocity for the same distance.* The mean distance corresponds to the points $C$ and $D$, the extremities of the minor axis, Fig. 102. It follows that the velocity at any point in the perihelion half $CAD$ of the orbit is greater than the circular velocity for the distance, namely, $\sqrt{\frac{\mu}{r}}$, while in the aphelion half $CBD$ it is less than the circular velocity for the distance.

As a further consequence, we infer that the mean distance of the orbit described by a body projected from a given point in any direction, will be less than, equal to or greater than the distance of projection, according as the velocity of projection is less than, equal to, or greater than the circular velocity belonging to that distance. But when it is equal to $\sqrt{2}$ times that velocity (see Art. 422), the mean distance becomes infinite and the orbit is a parabola.

## The Periodic Time.

**426.** Since $h$ is double the area swept over by the radius vector in a unit of time, if $T$ denotes the period of a complete revolution in an elliptic orbit, which is called *the periodic time*, $hT$ will be double the area of the ellipse. The area of the ellipse is $\pi ab$, where $b$, the minor semi-axis, equals $a\sqrt{(1 - e^2)}$, hence

$$hT = 2\pi a^2 \sqrt{(1 - e^2)},$$

and, since by equation (3), Art. 423, $h = \sqrt{\{a\mu(1 - e)^2\}}$,

$$T = \frac{2\pi}{\sqrt{\mu}} a^{\frac{3}{2}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Thus the periodic time depends only upon the mean distance. Solving this equation for $\mu$, we have

$$\mu = 4\pi^2 \frac{a^3}{T^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which gives the intensity of the force, or force upon a unit mass at a unit of distance, when the mean distance and periodic time of an elliptic orbit are known.

## Kepler's Laws.

427. The following laws with respect to the planetary motions were deduced by Kepler from a great mass of observations made by Tycho Brahe, combined with his own conjecture, regarding the variable distances of the planets.

1. The straight line joining a planet with the sun describes equal areas in equal times.
2. The planets describe ellipses having the sun at a focus.
3. The squares of the periodic times are proportional to the cubes of the mean distances.

Kepler was not possessed of correct notions regarding the nature of motion and force, but we have seen in Art. 397 how Newton, upon the basis of the true laws of motion, derived from the first of Kepler's laws the fact that the force acting upon the planets is directed toward the sun. From the second he showed (compare Art. 401) that the force acting upon any one planet varies inversely as the square of the distance. Finally, he showed that it follows from the third law, by means of the result expressed in equation (2) of the preceding article, that, regarding the sun as a fixed centre of force, the same law of variation with the distance governs its action upon the several planets. But, as we shall see hereafter, a slight modification of Kepler's third law is due to the fact that in each case the sun is not a fixed centre of force, but, like the planet itself, is free to move under the mutual attraction of the two bodies.

## Time of Describing a Given Arc of the Orbit.

**428.** The relation between $r$ and $\theta$, or equation of the orbit, given in equation (1), Art. 423, involves the constants of integration $h$ and $e$, and since, by equation (3) of the same article,

$$h^2 = a\mu(1 - e^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

it becomes

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when the constants employed are $a$ and $e$, the mean distance, or major semi-axis, and the eccentricity. The complete solution of the problem involves, in addition, the relation between $\theta$ and $t$, which is the integral of

$$r^2\frac{d\theta}{dt} = h,$$

when $r$ is the function of $\theta$ expressed in equation (2). This relation, which, as we have seen, is equivalent to the condition that double the area described by the radius vector in a unit of time shall be constantly equal to $h$, may, in the case of elliptic motion, be most conveniently derived by the geometrical process given in the following articles.

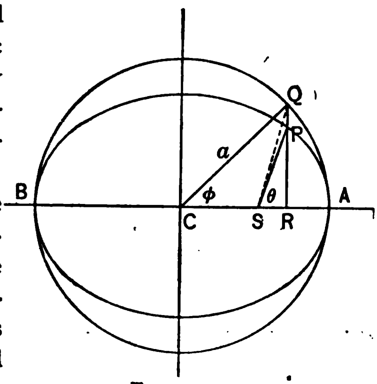

FIG. 103.

**429.** Let $P$, Fig. 103, be the position of the planet in its orbit, and produce the ordinate $PR$ to meet in $Q$ the circle described upon the major axis as a diameter. Join $PS$, $QS$ and $QC$. We shall use the eccentric angle $ACQ$ or $\phi$ as an auxiliary variable. It is called in Astron-

omy *the eccentric anomaly*, the vectorial angle $\theta$ at the focus $S$ being *the true anomaly.* We shall first express the time $t$ in terms of the eccentric anomaly.

Taking as the origin of time the instant when the planet is at the perihelion $A$, the principle of equable description of areas gives

$$ht = 2 \text{ area } ASP.$$

From a familiar property of the ellipse, the ratio $PR : QR$ is constant and equal to the ratio $b : a$; therefore the areas $PRA$ and $QRA$, as well as the triangles $PRS$, $QRS$, are in the same ratio, whence

$$ht = 2\,\frac{b}{a} \text{ area } QSA,$$

or

$$\frac{ah}{b}\,t = 2 \text{ sector } QCA - 2 \text{ triangle } QCS.$$

Now the area of the sector is $\frac{1}{2}a^2\phi$, and, since $CS = ae$, that of the triangle is $\frac{1}{2}a^2e \sin \phi$; hence

$$\frac{h}{ab}\,t = \phi - e \sin \phi.$$

Putting $n$ for the coefficient of $t$, so that by equation (1)

$$n = \frac{h}{ab} = \frac{\sqrt{\{a\mu\,(1 - e^2)\}}}{a^2\sqrt{(1 - e^2)}} = \frac{\sqrt{\mu}}{a^{\frac{3}{2}}}, \quad . \quad . \quad . \quad (3)$$

this relation is usually written

$$nt = \phi - e \sin \phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In this equation $nt$ may be regarded as the circular measure of an angle (not represented in the diagram) which is proportional to the time, and which assumes the values 0, $\pi$, $2\pi$, etc., at the instants when $\phi$ assumes the same values, and, therefore, as the figure shows, when $\theta$ assumes these values.

Accordingly, $nT = 2\pi$, where $T$ is the periodic time, as in Art. 426. It follows that $n$ is *the mean angular velocity* of the planet in its revolution about the sun. The angle $nt$ is called *the mean anomaly*. Equation (4) thus gives the value of the mean in terms of the eccentric anomaly.

**430.** The relation between the latter and the true anomaly is readily derived from the figure; since $CR = CS + SR$, we have

$$a \cos \phi = ae + r \cos \theta, \quad \ldots \ldots \quad (5)$$

whence, eliminating $r$ by equation (2),

$$\cos \phi = e + \frac{(1 - e^2) \cos \theta}{1 + e \cos \theta} = \frac{e + \cos \theta}{1 + e \cos \theta}. \quad . \quad . \quad (6)$$

The relation between $\phi$ and $\theta$ is, however, expressed more conveniently for computation by means of the functions of the half-angles. Thus, from

$$\tan^2 \tfrac{1}{2}\phi = \frac{1 - \cos \phi}{1 + \cos \phi},$$

we derive, by equation (6),

$$\tan^2 \tfrac{1}{2}\phi = \frac{1 + e \cos \theta - e - \cos \theta}{1 + e \cos \theta + e + \cos \theta} = \frac{(1 - e)(1 - \cos \theta)}{(1 + e)(1 + \cos \theta)},$$

whence

$$\tan \tfrac{1}{2}\phi = \sqrt{\frac{1 - e}{1 + e}} \tan \tfrac{1}{2}\theta. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For a given value of $\theta$, $\phi$ is found by this equation, and then $t$ is determined by means of equation (4).

**431.** Again, for the explicit expression of $t$ as a function of $\theta$, we derive, from equation (6),

$$\sin^2 \phi = \frac{(1 + e \cos \theta)^2 - (e + \cos \theta)^2}{(1 + e \cos \theta)^2} = \frac{(1 - e^2)(1 - \cos^2 \theta)}{(1 + e \cos \theta)^2}$$

whence

$$\sin \phi = \frac{\sqrt{(1 - e^2)} \sin \theta}{1 + e \cos \theta}.$$

Therefore, eliminating $\phi$ from equation (4),

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left[ 2 \tan^{-1} \sqrt{\frac{1 - e}{1 + e}} \tan \tfrac{1}{2}\theta - \frac{e \sqrt{(1 - e^2)} \sin \theta}{1 + e \cos \theta} \right], \quad (8)$$

This is the integral required in Art. 428, when the arbitrary constant, or epoch (see Art. 414), is determined by the condition that $t = 0$ when $\theta = 0$. Compare Int. Calc., Art. 36.

Equations (2) and (8) express $r$ and $t$ explicitly in terms of $\theta$; but it is impossible to express $r$ and $\theta$ explicitly in terms of $t$ by means of the elementary functions. The determination of their values for given values of $t$ (as required in the formation of an *ephemeris*, or table of daily positions of the planet) is known as

*Kepler's Problem.* Lagrange's solution of this problem consists in determining values of the eccentric anomaly $\phi$ corresponding to the given values of the mean anomaly $nt$, from equation (4), by means of Lagrange's Theorem (Diff. Calc., Art. 424); and then determining the values of $\theta$ by equation (7), and those of $r$ by the equation

$$r = a\,(1 - e\cos\phi)$$

derived from equations (2) and (5).

## EXAMPLES. XXI.

1. A triangle $AOB$, of which the sides $OA$, $OB$, and the angle at $O$ are the $a$, $b$ and $\alpha$ of Art. 389, revolves uniformly about $O$, so that $OA$ makes the angle $nt$ with the axis of $x$, and carries a circle of which $AB$ is a diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe the orbit represented in Fig. 100. Thence show that the axes of the ellipse are

$$\sqrt{(a^2 + b^2 + 2ab\cos\alpha)} \pm \sqrt{(a^2 + b^2 - 2ab\cos\alpha)}.$$

2. Show that the lines joining the points of contact in Fig. 100 are parallel to the diagonals of the rectangle, and verify for these points of the orbit that the velocities are inversely as the perpendiculars upon the tangents, in accordance with Art. 396.

3. Show that when $\alpha = \frac{1}{2}\pi$ in the equations of Art. 389, the component harmonic motions have the same phase, and the particle describes a diagonal of the rectangle in Fig. 100. Show also that, in the general case, the particle crosses this diagonal when $t$ is half the excess of the phase of the motion in $x$ over that of the motion in $y$.

4. Show that the velocity in the elliptical orbit of Art. 391 is proportional to the semi-diameter parallel to its direction, so that

the orbit is its own hodograph. Show also that $h = nab$, and derive thence the periodic time.

5. Find the radii of curvature at the vertices of the orbit of Ex. 4.

$$\frac{b^2}{a}; \frac{a^2}{b}.$$

6. Determine the law of attraction under which a body can describe a circle passing through the centre of force.

$$P = \frac{\mu}{r^5}.$$

7. In the case of the repulsive force $F = \mu r$, show that the orbit is an hyperbola whose centre is the centre of force.

8. If a body describing an ellipse under an attraction directly as the distance, enters a smooth tube at the extremity of an axis, how far will it go, and in what time?

9. A particle is attracted to one fixed centre, and repelled by another of equal intensity, each force varying directly as the distance. Show that it describes a parabola.

10. Show that a body acted upon, by any number of forces proportional to the distances, directed to or from fixed centres is an ellipse, or an hyperbola, according as the algebraic sum of the intensities is equivalent to an attraction or to a repulsion.

11. If the equation of a central orbit is of the form $u^2 = F(\theta)$, show that the force is proportional to

$$r\,(2FF'' - F'^2 + 4F^2).$$

12. In the orbit of Art. 391, show that the particle has the circular velocity corresponding to its distance when it is at the extremity of one of the equal pair of conjugate diameters.

13. Eliminating $\theta$ between equation (1) and (2), Art. 398, show that the integral of the result is a form of the equation of energy.

14. Let the diameter of the orbit in Ex. 6 be $OA = a$. Show that the velocity in the orbit is that due to infinity, and find its value at $A$; find also the periodic time.

$$\frac{\sqrt{\mu}}{a^2\sqrt{2}}; \frac{\pi a^3}{\sqrt{(2\mu)}}.$$

15. Show that the kinetic energy at $A$ in Ex. 14 is one half of that possessed by a body having the "circular velocity" at $A$.

16. Determine the orbit of a body under the action of a force varying inversely as the $n$th power of the distance, the velocity being that due to infinity.

$$r^{\frac{n-3}{2}} = \frac{\sqrt{(2\mu)}}{h\sqrt{(n-1)}} \cos \frac{n-3}{2}\theta.$$

17. Show that, with the law of force in Ex. 16, the circular velocity at a given distance bears to the velocity from infinity at the same distance the fixed ratio $\sqrt{(n-1)} : \sqrt{2}$. Show also that the circular orbit is stable when $n = 2$, and unstable when $n > 3$.

*In the following examples the law of attraction is that of gravity, namely,* $P = \mu u^2$.

18. Express the function $\psi(u)$ in terms of $e$ and $h$, and thence obtain the apsidal distances.

$$h^4\psi(u) = \mu^2 (e^2 - 1) + 2\mu h^2 u - h^4 u^2.$$

19. Determine the radii of curvature at the apsides and at the extremity of the minor axis by means of the normal acceleration.

20. Show that at the mean distance the kinetic energy of the body is a mean proportional between its extreme values, and that at the point where $\theta = 90°$ it is an arithmetical mean between the same values.

21. A body is projected with the velocity $V$ in a direction making the angle $\beta$ with the prime vector, upon which the point of projection is situated at the distance $R$ from the centre of force. Prove that, $\alpha$ being the vectorial angle of the perihelion,

$$e^2 = \cos^2\beta + \sin^2\beta\left[\frac{V^2R}{\mu} - 1\right]^2,$$

and

$$e\cos\alpha = \frac{V^2R\sin^2\beta}{\mu} - 1.$$

22. From the relation between the mean and eccentric anomalies, Art. 429, show that the time of falling from rest on the

circle of total energy, that is from the distance $2a$, to the distance $r$ from the centre of force, is

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left[\cos^{-1}\frac{r-a}{a} + \frac{\sqrt{(2ar-r^2)}}{a}\right].$$

Compare equation (5), Art. 347.

23. Show that, for a central orbit, the hodograph is the curve inverse to the pedal from the centre of force turned through 90°. Thence show that the hodograph of the planetary motion is a circle.

# CHAPTER XI.

## MOTION OF RIGID BODIES.

### XXII.

### Action of Inertia in Rotation.

**432.** We have seen in Art. 289 that, in motions of translation of a rigid body, the resultant of the inertia forces acts at the centre of inertia; so that, when there are no external forces acting except those of gravity (of which the resultant acts at the same point), the body may be treated as a particle. In this chapter we shall consider the action of inertia in other kinds of motion, and of external forces applied at points other than the centre of inertia.

Let us first suppose the rigid body to admit of no motion except rotation about a fixed axis. A perpendicular of indefinite length drawn from a point of the axis in the substance of the body generates in the rotation a plane. Let $\theta$ be the angle which this perpendicular makes at the time $t$ with a fixed direction in the plane; then

$$\omega = \frac{d\theta}{dt}$$

is called *the angular velocity* of the rotation. The unit of angular velocity is of course the angle whose arcual measure is unity,

sometimes called *the radian.* The linear velocity of a point at a distance $r$ from the axis is

$$v = \frac{ds}{dt} = \frac{rd\theta}{dt} = r\omega.$$

When the angular velocity is constant, the rotation is said to be *uniform;* every particle has uniform circular motion, and, denoting its mass by $m$ and its distance from the axis by $r$, the inertia of the particle is simply its centrifugal force $m\omega^2 r$ (Art. 359). Since the centrifugal force of each particle acts in a line passing through the axis, the resultant of the whole inertia will be balanced by the resistance of the axis. Hence, if the axis be smooth, there will be no resistance to uniform rotation.

**433.** If the angular velocity is not constant, every particle at a distance $r$ from the axis will have the tangential acceleration

$$\frac{dv}{dt} = r\frac{d\omega}{dt} = r\frac{d^2\theta}{dt^2}.$$

Hence, $m$ being the mass of the particle, it exerts a tangential force of inertia equal to

$$mr\frac{d^2\theta}{dt^2}.$$

This component of inertia resists change in the angular speed of rotation, hence its efficiency must be estimated (like that of a force in producing rotation, Art. 93) by means of its moment about the axis. The arm with which the tangential inertia acts is $r$, hence its moment is

$$mr^2\frac{d^2\theta}{dt^2},$$

and, since the normal inertia has no moment about the axis, this is the whole moment of the inertia of $m$.

The whole moment resisting the rotation of the body is found by summing the expressions of this form for all the particles of the body; that is to say, it is

$$\Sigma mr^2 \cdot \frac{d^2\theta}{dt},$$

since the factor $\frac{d^2\theta}{dt^2}$, which is *the angular acceleration*, is common to all the expressions.

## Moments of Inertia.

434. The moment of the impressed force (or, if more than one force is acting, the resultant moment of the impressed forces) which produces an angular acceleration is equal to the moment of the inertia which resists it. Denoting the former by $K$, and putting $I$ for the factor $\Sigma mr^2$ in the expression found above for the moment of inertia, we have, therefore, the equation

$$K = I\frac{d^2\theta}{dt^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

which enables us, when $I$ has been found for the given body and axis, to determine the angular acceleration which will be produced by a given force or system of forces.

It has become customary to call the factor $I$ the *moment of inertia*, although properly the second member of equation (1) is the moment of inertia, or *rotational inertia*. The factor $I$, which is analogous to the mass in the formula for linear acceleration,

$$F = M\frac{d^2s}{dt^2},$$

is only the mass-factor of the rotational inertia, just as $M$ is that of the inertia of translation, the other factor being in each case an acceleration.

435. Let $M = \Sigma m$ be the whole mass of the body whose moment of inertia is $I = \Sigma mr^2$. If the particles were all at a common distance, $r = a$, from the axis, we should have $I = a^2 M$. For example, in the case of a heavy fly-wheel revolving about an axis passing through its centre and perpendicular to its plane, the mass may, with very little error, be assumed to be concentrated into the circumference of a circle whose radius $a$ is the mean radius of the fly-wheel. Now suppose that a force $P$ is applied at the circumference of an axle whose radius is $b$. The moment of the applied force is $K = bP$, and the moment of inertia of the fly-wheel is $Ma^2$. Hence, substituting in equation (1) of the preceding article, we have

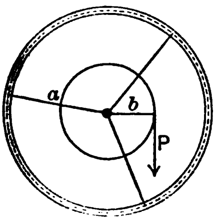

FIG. 104.

$$bP = a^2 M \frac{d^2\theta}{dt^2}.$$

For instance, if the radius of the fly-wheel is 3 feet, its weight 100 pounds, the radius of the axle 6 inches, and the applied force 20 pounds, we have $K = \frac{1}{2} \times 20 = 10$ pounds-feet and $I = 3^2 \times \dfrac{100}{g}$; therefore

$$10 = \frac{900}{32} \frac{d^2\theta}{dt^2};$$

whence $\dfrac{d^2\theta}{dt^2} = \dfrac{16}{45}$. Thus, the given moment produces in this wheel an angular acceleration of $\frac{16}{45}$ radians. That is to say, if it acted *for one second* upon the fly-wheel, originally at rest, it would produce an angular velocity of $\frac{16}{45}$ (which is about .113 of one revolution) per second.

The linear velocity acquired by each point of the rim is in this case $2\frac{2}{15}$ f/s.

## Moment of Inertia of a Continuous Body.

436. In the expression $I = \Sigma mr^2$, the mass is regarded as made up of separate parts treated as particles, each particle having its special value of $r$. For a continuous mass we must (as in the case of statical moments) replace $m$ by $dM$, an element of mass, and the sign of summation by that of integration. Thus we write

$$I = \int r^2 dM,$$

in which $dM$ is an element of mass at the distance $r$ from the axis of rotation. If we can express the entire element of mass at the distance $r$ in terms of $r$, we can find $I$ by a single integration. Suppose, for example, we have to find the moment of inertia of a homogeneous cylinder of length $l$ and radius $a$ about its geometrical axis. The entire element at the distance $r$ from the axis is the mass of an element of volume of thickness $dr$ and having a cylindrical surface of radius $r$ and length $l$. Then, denoting the uniform density by $\rho$, we have

$$dM = \rho . 2\pi rl dr.$$

Substituting in the expression for $I$, we find

$$I = 2\pi\rho l \int_0^a r^3 dr = \frac{\pi\rho l a^4}{2}.$$

## The Radius of Gyration.

437. The moment of inertia of a particle of mass $M$ at a distance $k$ from the axis is $k^2M$; hence, if we put

$$I = k^2M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$k$ is the radius of a circumference upon which if the whole mass were concentrated (as in the illustration of the fly-wheel, Art. 435), it would have the same moment of inertia that it actually has.

Thus $k$ may be regarded as the radius of the equivalent fly-wheel; it is called *the radius of gyration* of the body for the given axis.

Equation (1) when written in the form

$$\Sigma r^2 m = k^2 \Sigma m$$

shows that $k^2$ is the average value of the squared distance of the particles from the axis. When, as is usually the case, the value of $M$ is known, we need only, in questions involving the moment of inertia, to know the value of $k^2$, which, being simpler than that of $I$, is more easily remembered. Thus, in the example of the preceding article, $M$ is known from the known volume of a cylinder, namely,

$$M = \pi \rho l a^2.$$

Hence, from the value of $I$ found above, we have

$$k^2 = \tfrac{1}{2} a^2$$

for the squared radius of gyration of a homogeneous cylinder about its geometrical axis.

### Interaction of Inertia in Rotation and Translation.

**438.** When a mutual action exists between two bodies, one having a motion of translation and the other one of rotation, their accelerations and mutual action may be found by a method similar to that employed in Art. 311.

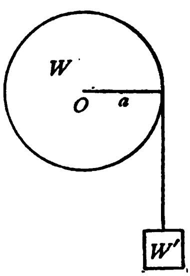

FIG. 105.

As an illustration, suppose a homogeneous cylinder of weight $W$ and radius $a$, Fig. 105, mounted on a smooth horizontal axis, to have a fine string wound about it, to the free end of which a weight $W'$ is attached. Let us find the acceleration, and the tension, $T$, of the string. Denoting the space through which $W'$ falls by $s$, and the angle through which the cylinder turns by $\theta$, we have $s = a\theta$, whence the

linear and angular accelerations are connected by the relation

$$\frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

The moment of inertia of a cylinder whose mass is $M$ is found above to be $I = \frac{1}{2}a^2M$, and the impressed moment is $K = aT$; hence, by equation (1), Art. 434, the kinetic equilibrium of the cylinder gives

$$aT = \frac{1}{2}a^2M\frac{d^2\theta}{dt^2},$$

or

$$T = \frac{W}{2g}\frac{d^2s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In like manner, that of $W'$ gives

$$W' - T = \frac{W'}{g}\frac{d^2s}{dt^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding, to eliminate $T$,

$$W' = \frac{W + 2W'}{2g}\frac{d^2s}{dt^2};$$

whence

$$\frac{d^2s}{dt^2} = \frac{2gW'}{W + 2W'},$$

and, substituting in equation (1),

$$T = \frac{WW'}{W + 2W'}.$$

## The Energy of Rotation.

439. The kinetic energy of a body rotating about a fixed axis is the sum of the kinetic energies of its particles. The linear velocity of a particle of mass $m$ at the distance $r$ from the axis is

$r\omega$, and therefore $\frac{1}{2}mr^2\omega^2$ is its kinetic energy. Hence, the whole kinetic energy of rotation is

$$\tfrac{1}{2}\Sigma mr^2\omega^2 = \tfrac{1}{2}I\omega^2,$$

in which the quantity $I$ again appears as analogous to $M$ in the corresponding expression involving the velocity of translation, namely, $\frac{1}{2}Mv^2$.

Using this expression, we may apply the principle of work and energy directly to questions involving spaces and velocities. For example, in the illustration of the preceding article, to find the velocity acquired when $W'$ falls from rest through the space $s$: Denoting this velocity by $v$, the angular velocity of the cylinder is $\omega$, where $v = a\omega$. The work done by gravity is $W's$, and this work produces kinetic energy in each of the bodies. That of $W$ is $\frac{1}{2}I\omega^2$, where $I = \frac{1}{2}a^2M$; hence,

$$\text{kinetic energy of } W = \frac{W}{4g}a^2\omega^2 = \frac{W}{4g}v^2,$$

$$\text{kinetic energy of } W' = \frac{W'}{2g}v^2.$$

Therefore

$$\frac{W + 2W'}{4g}v^2 = W's,$$

or

$$v^2 = \frac{4W'gs}{W + 2W'}.$$

This velocity is, of course, the same that would be found by treating $W'$ as a body moving with the constant acceleration found in the preceding article.

### Work done in an Angular Displacement.

440. Let a constant force $P$ act upon a body free to rotate about a fixed axis, as in Fig. 104, p. 316, the force acting with a constant arm $b$ so as to have a constant moment $K = Pb$. In

an angular displacement through the angle $\theta$ the force works through the space $b\theta$, equal to the arc of the axle from which the string is unwound. Hence the work done is $Pb\theta$, or $K\theta$. That is to say, *the work done in an angular displacement is the product of the turning moment acting and the angular displacement.*

It will be noticed that the latter factor is an abstract number or ratio, and accordingly the units of work and of moment are the same, namely, the foot-pound or pound-foot.

**441.** The equation of rotary motion about a fixed axis is, by Art. 434,

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{K}{I},$$

where

$$\omega = \frac{d\theta}{dt}.$$

Eliminating $dt$, after the analogy of Art. 291, we have

$$\omega d\omega = \frac{K}{I} d\theta.$$

The integral of this between limits is the equation of energy,

$$\tfrac{1}{2}I(\omega_2^2 - \omega_1^2) = \int_{\theta_1}^{\theta_2} K d\theta.$$

The second member (in which $K$ may be a function of $\theta$) expresses the work done by $K$, while $\theta$ varies from $\theta_1$ to $\theta_2$; hence the equation shows that this work is equal to the kinetic energy gained, as in the corresponding equation of Art. 294.

## Moment of Inertia of a Geometrical Magnitude.

**442.** In the case of a homogeneous solid of density $\rho$ and volume $V$, the mass is $M = \rho V$, and the moment of inertia of this

mass is $k^2M = k^2\rho V$. Omitting the constant factor $\rho$, the quantity

$$I = k^2 V$$

is called *the moment of inertia of the volume* with respect to a given axis, just as, in Art. 181, the product $\bar{x}V$ is called the statical moment of the volume with respect to a certain plane.

In like manner, if $k$ is the radius of gyration of a mass regarded as concentrated with uniform density into a given surface of area $A$,

$$I = k^2 A$$

is called the moment of inertia of the area. Again, if $k$ is the radius of gyration of a mass regarded as concentrated with uniform density into a line of length $s$,

$$I = k^2 s$$

is called the moment of inertia of the line.

443. In the case of the line, the expression for $I$ as an integral is

$$I = \int r^2 ds,$$

where $r$ is the distance of the element $ds$ from the axis. This expression involves but a single integration, of which the limits are the values of $s$ at the two extremities of the line. For example, let us find the moment of inertia of a line of length $a$ about an axis perpendicular to it passing through one end. Taking this end as origin, the element is $dx$, and its distance from the axis is $x$; hence

$$I = \int_0^a x^2 dx = \frac{a^3}{3} = \frac{a^2}{3} \cdot a.$$

In the last member we have written $I$ in the form $k^2 s$, so that $k^2 = \frac{1}{3}a^2$ is the squared radius of gyration.

The displacement in a direction parallel to the axis of any portion of the mass which is supposed concentrated in the line evidently cannot change the moment of inertia; therefore $\frac{1}{3}a^2$ is also the squared radius of gyration of a rectangle whose sides are $a$ and $b$ about a side of length $b$.

444. As another example, let us find the radius of gyration of the arc in Fig. 60, p. 131, about the axis of $x$. The element $ds$ is at the distance $y$ from the axis; and, expressing $ds$ and $y$ in terms of $\theta$, we have $ds = ad\theta, y = a \sin \theta$. Hence

$$I = a^2 \int_{-\alpha}^{\alpha} \sin^2\theta \, d\theta = a^2(\alpha - \sin\alpha\cos\alpha).$$

Dividing by $s$, which is $2a\alpha$,

$$k^2 = \frac{a^2}{2}\left(1 - \frac{\sin\alpha\cos\alpha}{\alpha}\right).$$

When $\alpha = 0$, we have $k^2 = 0$; when $\alpha = \frac{1}{2}\pi$, we have $k^2 = \dfrac{a^2}{2}$ for the radius of gyration of a semicircumference about the bisecting diameter. When $\alpha = \pi$, the same value is found for the complete circumference about a diameter. This obviously should be the case, because both the moment of inertia and the length have now double the values which correspond to the semicircumference.

## The Moment of Inertia of a Plane Area.

445. In finding the moment of inertia of a plane area about an axis in its plane, we shall suppose its curved boundaries to be referred to rectangular coordinate axes.

For example, let us find the radii of gyration of an ellipse about its axes. The equation of the ellipse referred to its axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

To find the moment of inertia about the axis of $y$ by a single integration, we take for the element of area $2ydx$, as in the diagram. Since all points of this element are at the same distance $x$ from the axis of $y$, the moment of inertia about that axis is

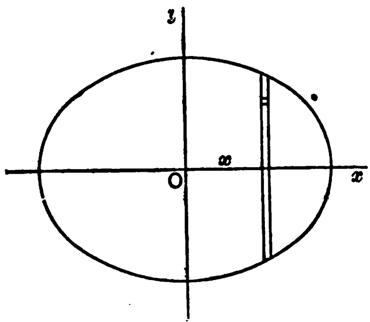


FIG. 105.

$$I = 2\int_{-a}^{a} yx^2 dx.$$

Substituting the value of $y$ from the equation of the curve,

$$I = 2\frac{b}{a}\int_{-a}^{a} \sqrt{(a^2 - x^2)}x^2 dx;$$

and, putting $x = a \sin \theta$, this becomes

$$= 4a^3b\int_0^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta\, d\theta = 4a^3b\frac{1 \cdot 1}{4 \cdot 2}\frac{\pi}{2} = \frac{\pi a^3 b}{4}.$$

Since the area of the ellipse is $A = \pi ab$, we have $k^2 = \frac{1}{4}a^2$.

446. To illustrate the employment of double integration, we shall apply it to find the moment of inertia of this ellipse about the axis of $x$. The element of area is now the point-element

$$d^2A = dx\, dy$$

situated at the point $(x, y)$, and its moment of inertia about the axis of $x$ is

$$d^2I = y^2 dx\, dy.$$

The two integrations required may be performed in either order. If we perform the $x$-integration first, we shall be summing up the elements along a line parallel to the axis of $x$, and the remaining part of the process will be the same as that of the preceding article, with an interchange of the coördinates $x$ and $y$ and the

constants $a$ and $b$. But if we perform the $y$-integration first, we obtain an integral of different form. Thus

$$dI = dx\int_{-y_1}^{y_1} y^2 dy = \tfrac{2}{3}y_1^3 dx,^*$$

where $y_1$ is the ordinate of the ellipse. Hence, substituting and integrating,

$$I = \frac{4b^3}{3a^3}\int_0^a (a^2 - x^2)^{\frac{3}{2}} dx$$

$$= \frac{4b^3a}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{4b^3a}{3}\,\frac{3\cdot 1}{4\cdot 2}\,\frac{\pi}{2} = \frac{\pi b^3 a}{4}.$$

Hence, for this axis, $k^2 = \frac{1}{4}b^2$, agreeing with the result previously found.

## The Polar Moment of Inertia of an Area.

447. The general expressions for the moment of inertia of an area about the axes of $x$ and $y$ respectively are

$$I_x = \Sigma my^2, \qquad I_y = \Sigma mx^2.$$

Consider now the moment of inertia of the area about an axis passing through the origin and perpendicular to the plane. If $r$ denote the distance of the particle $m$ situated at the point $(x, y)$ from the origin, $\Sigma mr^2$ is the required moment of inertia, which may be denoted by $I_z$. Now, since

$$r^2 = x^2 + y^2,$$

* In this process we have summed up the point elements along a line parallel to the axis of $y$. Accordingly we have obtained the moment of inertia of the element drawn in Fig. 105. Since the radius of gyration of this element about the axis of $x$ is obviously the same as that of the ordinate $y$, if we take the result of Art. 443 as known, this value of $dI$ can be obtained by multiplying $dA$, which is $2ydx$, by its squared radius of gyration.

we have, by summation,

$$\Sigma mr^2 = \Sigma mx^2 + \Sigma my^2.$$

Hence we have, for any area,

$$I_z = I_x + I_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The moment of inertia $I_z$ about an axis perpendicular to the plane is often called a *polar moment of inertia.* Equation (1) then shows that the *polar moment of inertia of an area about a given axis is equal to the sum of the moments of inertia about any pair of axes in the plane which intersect the polar axis, and are at right angles to each other.*

448. Dividing equation (1) by the mass $M$, we derive

$$k_z^2 = k_x^2 + k_y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For example, from the results found in Arts. 445, 446 we find, for the squared radius of gyration of an ellipse about an axis through its centre and perpendicular to its plane,

$$k_z^2 = \frac{a^2 + b^2}{4}.$$

The theorem expressed by equation (1) or equation (2) gives usually the best method of finding a polar radius of gyration; but if this radius is known we may sometimes use the theorem to find the radius of gyration for an axis in the plane. For example, the process of Arts. 436, 437 is equivalent to showing that the value of $k^2$ for a circle about its geometrical axis is $k_z^2 = \frac{1}{2}a^2$. Taking this as known, and noticing that for the circle $k_x$ and $k_y$ are equal by symmetry, we have

$$k_z^2 = \tfrac{1}{2}a^2 = 2k_x^2,$$

giving $k_x^2 = \frac{1}{4}a^2$, for the circle about a diameter.

449. Again, we found in Art. 443 that, for a square of side $a$, the squared radius of gyration about a side is $\frac{1}{3}a^2$. Hence, by equa-

tion (2), we have, for an axis through a vertex and perpendicular to the plane, $k^2 = \frac{2}{3}a^2$. Now passing to a square of side $2a$ and a polar axis through its centre, we have for this also $k^2 = \frac{2}{3}a^2$, because we have multiplied by four both the area and the moment of inertia. Furthermore, take *any* two axes in the plane, the moments of inertia about them are equal by symmetry. Hence the squared radius of gyration for one of them is one-half that found above, namely $\frac{1}{3}a^2$. This is therefore the squared radius of gyration, for the square whose side is $2a$, about *any* axis in its plane passing through its centre.

## Employment of Polar Coordinates.

**450.** When the boundary of the area is given by its polar equation, the ultimate or point element of area

$$d^2A = r\,dr\,d\theta$$

should be employed. Its distance from the initial line or axis of $x$ is then $r \sin\theta$, that from the axis of $y$ is $r\cos\theta$, and that from the axis of $z$ is $r$. For example, we may thus find the moment of inertia of the circle in Fig. 107 about the tangent $Oy$. The polar equation of the circle referred to the pole $O$ upon its circumference, the diameter being the initial line, is

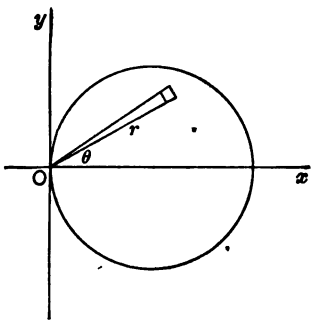

FIG. 107.

$$r_1 = 2a\cos\theta.$$

The moment of the element $d^2A$ about $Oy$ is

$$d^2I_y = r^3\cos^2\theta\,dr\,d\theta.$$

Whence

$$I_y = 2\int_0^{\frac{\pi}{2}}\int_0^{r_1} r^3 dr\cos^2\theta\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} r_1^4\cos^2\theta\,d\theta.$$

Substituting the value of $r_1$ from the equation of the circle,

$$I_y = 8a^4 \int_0^{\frac{\pi}{2}} \cos^6 \theta \, d\theta = 8a^4 \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \frac{\pi}{2} = \frac{5\pi a^4}{4} = \frac{5a^2}{4} A.$$

### The Moment of Inertia of a Solid.

**451.** The moment of inertia of any solid of revolution about its geometrical axis can be found, as in Art. 436, by means of a single integration. In general, the length of the cylindrical element is variable, and must be expressed in terms of its radius. For example, in the case of the cone of which Fig. 64, p. 145, is a section through the geometrical axis, $y$ is the radius, and $a - x$ the length of the cylindrical element. Hence the volume of the element is

$$dV = 2\pi y(a - x)dy ;$$

whence

$$dI = 2\pi y^3(a - x)dy.$$

Substituting the value $x = \frac{a}{b}y$, and integrating,

$$I = \frac{2\pi a}{b} \int_0^b (by^3 - y^4)dy = 2\pi ab^4(\tfrac{1}{4} - \tfrac{1}{5}) = \frac{\pi ab^4}{10} = \frac{3b^2}{10} V.$$

**452.** We can express the amount of inertia by a single integral also when the polar moment of inertia of the section of the solid perpendicular to the axis is known. For example, the equation of the ellipsoid referred to its rectangular axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 ;$$

required to find the moment of inertia about the axis of $z$. The

section parallel to the plane of $xy$ at the distance $z$ from that plane is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{c^2-z^2}{c^2},$$

of which the semi-axes are

$$\alpha=\frac{a}{c}\sqrt{(c^2-z^2)} \quad \text{and} \quad \beta=\frac{b}{c}\sqrt{(c^2-z^2)}.$$

Now the area of this ellipse is $\pi\alpha\beta$; and by Art. 448, its polar squared radius of gyration is $\frac{1}{4}(\alpha^2+\beta^2)$. Hence we have, for the element of volume,

$$dV=\pi\frac{ab}{c^2}(c^2-z^2)dz,$$

and for the square of its radius of gyration about the axis of $z$

$$k^2=\frac{a^2+b^2}{4c^2}(c^2-z^2).$$

Therefore

$$dI=\frac{\pi ab(a^2+b^2)}{4c^4}(c^2-z^2)^2dz,$$

and

$$I=\frac{\pi ab(a^2+b^2)}{2c^4}\int_0^c(c^4-2c^2z^2+z^4)dz$$

$$=\frac{\pi ab(a^2+b^2)}{2c^4}(1-\tfrac{2}{3}+\tfrac{1}{5})c^5=\frac{4\pi abc}{15}(a^2+b^2).$$

Since the volume is $V=\frac{4}{3}\pi abc$, we have, denoting the radius of gyration about the axis of $z$ by $k_z$,

$$k_z^2=\tfrac{1}{5}(a^2+b^2);$$

and in like manner,

$$k_x^2 = \tfrac{1}{5}(b^2 + c^2),$$

$$k_y^2 = \tfrac{1}{5}(c^2 + a^2).$$

In particular, when the semi-axes are equal, we have, for the radius of gyration of the homogeneous sphere whose radius is $a$ about a diameter,

$$k^2 = \tfrac{2}{5}a^2.$$

### Separate Calculation of $\Sigma mx^2$, $\Sigma my^2$ and $\Sigma mz^2$.

**453.** We have seen in Art. 447 that, in the case of an area or a lamina referred to three rectangular axes, the axes of $x$ and $y$ being in the plane of the lamina,

$$I_z = \Sigma mx^2 + \Sigma my^2.$$

Now the moment of inertia of any particle $m$ about the axis of $z$ does not depend in any way upon the value of $z$. Hence this equation is also true for a solid of any form. But the moments of inertia about the other axes are now

$$I_x = \Sigma my^2 + \Sigma mz^2, \qquad I_y = \Sigma mz^2 + \Sigma mx^2;$$

so that we no longer have, as in the case of a lamina, one of the three moments about rectangular axes equal to the sum of the other two. On the contrary, in the general case of a solid, each of these moments is less than the sum of the other two.

**454.** We may often, with advantage, use these expressions in calculating the moment of inertia of a solid. For example, to express $\Sigma mz^2$ for the ellipsoid of Art. 452 as a single integral, we need only to know the volume of the element at the distance $z$

from the plane of $xy$. Thus, using the value of $dV$ employed in that article, we have

$$\Sigma mz^2 = \int_{-c}^{c} z^2 dV = \frac{2\pi ab}{c^2}\int_0^c (c^2z^2 - z^4)dz$$
$$= \frac{2\pi ab}{c^2}c^5(\tfrac{1}{3} - \tfrac{1}{5}) = \frac{4\pi abc^3}{15} = \frac{c^2}{5}V.$$

In like manner, we have

$$\Sigma mx^2 = \frac{a^2}{5}V, \qquad \Sigma my^2 = \frac{b^2}{5}V;$$

whence, by the equations of the preceding article,

$$I_x = \frac{b^2 + c^2}{5}V, \qquad I_y = \frac{c^2 + a^2}{5}V, \qquad I_z = \frac{a^2 + b^2}{5}V,$$

agreeing with the result found in Art. 452.

**455.** The moment of inertia of a spherical shell about a diameter can be found without integration by means of the equations of Art. 453. For the equation of the spherical surface referred to rectangular planes passing through the centre is

$$x^2 + y^2 + z^2 = a^2.$$

Hence, if $M$ is the mass of the spherical shell, or mass supposed to be uniformly concentrated on the spherical surface,

$$\Sigma mx^2 + \Sigma my^2 + \Sigma mz^2 = \Sigma ma^2 = a^2M.$$

But, by symmetry,

$$\Sigma mx^2 = \Sigma my^2 = \Sigma mz^2;$$

therefore the value of each of these quantities is $\frac{1}{3}a^2M$, and

$$I_x = \tfrac{2}{3}a^2M, \qquad \text{whence} \qquad k^2 = \tfrac{2}{3}a^2$$

is the squared radius of gyration about a diameter.

**456.** This result furnishes a convenient method of finding the moment of inertia of a sphere when the density is a function of the distance from the centre.

For example, to find the moment of inertia of the sphere considered in Art. 186, of which the weight per unit volume is $\dfrac{w_1 a^2}{r^2}$, and therefore the mass per unit volume is $\rho = \dfrac{w_1 a^2}{gr^2}$. Taking for element of volume the spherical shell of radius $r$ and thickness $dr$, we have

$$dV = 4\pi r^2 dr, \qquad dM = 4\pi\rho r^2 dr = \frac{4\pi w_1 a^2}{g} dr.$$

Multiplying by the value of $k^2$ for the shell, which is $\frac{2}{3}r^2$, we find

$$dI = \frac{8\pi w_1 a^2}{3g} r^2 dr.$$

Integrating from 0 to $a$, and using the mass as found in Art. 186, we obtain

$$I = \frac{8\pi w_1 a^2}{3g} \cdot \frac{a^3}{3} = \frac{4\pi w_1 a^3}{g} \cdot \frac{2a^2}{9};$$

whence for this sphere $k^2 = \frac{2}{9}a$.

## Selection of the Element of Integration.

**457.** The examples already given show that the mode of selecting the element depends chiefly upon the character of the bounding curve or surface, which determines the limits of integration. As a further illustration, consider the solid generated by revolving the circle in Fig. 106, Art. 450, about the axis of $y$. The most convenient element of volume is that generated in this rotation by the element of area $d^2A$. The path described by this element of area is the circumference of the circle whose radius is $r\cos\theta$; hence

$$d^2V = 2\pi r\cos\theta \, . \, rdrd\theta,$$

Now, to obtain the moment of inertia about the axis of $y$, we multiply this circular element* by the square of its radius of gyration, which is its own radius, $r \cos \theta$. This gives

$$d^2I = 2\pi r^4 dr \cos^3 \theta \, d\theta;$$

whence

$$I = 4\pi \int_0^{\frac{\pi}{2}} \int_0^{r_1} r^4 dr \cos^3 \theta \, d\theta = \frac{4\pi}{5} \int_0^{\frac{\pi}{2}} r_1^5 \cos^3 \theta \, d\theta.$$

Substituting $r_1 = 2a \cos \theta$, from the equation of the circle,

$$I = \frac{2^7 \pi a^5}{5} \int_0^{\frac{\pi}{2}} \cos^8 \theta \, d\theta = \frac{2^7 \pi a^5}{5} \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2} = \frac{7\pi^2 a^5}{2}.$$

By Pappus's Theorem the volume is $2\pi^2 a^3$; hence for this solid we have

$$k_y^2 = \frac{7a^2}{4}.$$

**458.** For the anchor-ring in general, it is simpler to refer the generating circle to its centre, because the limits of each variable will then be independent of the other. Thus, referring to Fig. 62, p. 136, let us find the moment of inertia of the anchor-ring about its axis $AB$. The radius of the circle described by the element of area $d^2A = rdrd\theta$ is

$$b + r \cos \theta,$$

and this is also the radius of gyration of the element of volume generated by the rotation of the element of area. Hence

$$d^2I = 2\pi(b + r \cos \theta)^2 \, rdrd\theta.$$

* It is the shape of the solid, and *not* the position of the axis about which the moment of inertia is required, which determines the element to be used. We shall see, in the next section, that the element of volume thus determined can be used in finding the moment of inertia about any axis, because we have means of finding the radii of gyration of elements of simple form about any axis.

The limits for $r$ are o and $a$, and the limits for $\theta$ are o and $2\pi$. Hence, expanding and performing the $r$-integration, we have

$$I = 2\pi\int_0^{2\pi}\left(b^3\frac{a^2}{2} + 3b^2\frac{a^3}{3}\cos\theta + 3b\frac{a^4}{4}\cos^2\theta + \frac{a^5}{5}\cos^3\theta\right)d\theta$$

$$= 2\pi\left[\frac{b^3a^2}{2}\cdot 2\pi + \frac{3ba^4}{4}\cdot\pi\right] = \frac{\pi^2ba^2}{2}(4b^2 + 3a^2).$$

By Pappus's Theorem the volume is $V = 2\pi^2ba^2$; hence, for the anchor-ring,

$$k^2 = b^2 + \frac{3}{4}a^2.$$

This gives the excess of the radius of gyration of a fly-wheel whose rim has a circular section over the mean radius $b$, which was, in Art. 435, taken as its approximate value when $a$ is small relatively to $b$.

### EXAMPLES. XXII.

1. Find the moment of inertia of a triangle of base $b$ and altitude $h$ about an axis passing through the vertex and parallel to the base. $A\cdot\frac{h^2}{2}$.

2. Find the moment of inertia of the same triangle about the base. $A\cdot\frac{h^2}{6}$.

3. If the altitude $h$ of a triangle divides the base into the segments $a$ and $b$, find the radius of gyration about the altitude. $k^2 = \frac{a^2 - ab + b^2}{6}$.

4. If $W$ in Art. 438 is a drum or hollow cylinder whose thickness may be neglected, show that the acceleration and tension are the same as if $W'$ dragged $W$ along a smooth horizontal table as in Fig. 87, p. 248.

5. Find the radius of gyration for the arc of the cycloid

$$x = a(\psi - \sin\psi), \qquad y = a(1 - \cos\psi),$$

about the axis of $x$. $k^2 = \frac{32}{15}a^2$.

6. Find the radius of gyration for the area of the cycloid in Ex. 5 about its base.
$$k^2 = \frac{35}{36}a^2.$$

7. Determine $k^2$ about the same axis for the surface of revolution generated by revolving the cycloid about its base.
$$k^2 = \frac{96}{35}a^2.$$

8. Determine the radius of gyration of the area of the lemniscata $r^2 = a^2 \cos 2\theta$ about a tangent at the origin.
$$k^2 = \frac{\pi a^2}{16}.$$

9. Find the radius of gyration of the area of the lemniscata about the axis of the curve.
$$\frac{a^2}{48}(3\pi - 8).$$

10. Determine the radius of gyration of a fly-wheel of mean radius $a$ when the rim has a rectangular section, $t$ being its thickness.
$$k^2 = a^2 + \tfrac{1}{4}t^2.$$

11. A uniform door, 3 feet wide, weighing 80 pounds, is swinging on its hinges, and the edge has a velocity of 8 feet per second. How many foot-pounds of energy must be expended in stopping it? $26\frac{2}{3}$.

12. Find the moment of inertia of a lamina in the form of a regular hexagon whose side is $a$ about one of its central diagonals.
$$\frac{5\sqrt{3}\,a^4}{16}.$$

13. Show that, if a solid of revolution is referred to rectangular axes, that of $x$ being the geometrical axis, $\Sigma mz^2 = \Sigma my^2 = \frac{1}{2}I_x$ (the density of the solid being assumed unity). Hence, using the value found for the cone in Art. 451 and determining $\Sigma mx^2$ independently, find the radius of gyration for a perpendicular to the geometrical axis passing through the vertex.
$$k^2 = \frac{3}{20}(b^2 + 4h^2).$$

14. Determine the radius of gyration of a paraboloid about its axis, the radius of the base being $b$ and the height $h$.
$$k^2 = \tfrac{1}{3}b^2.$$

15. Determine by the method suggested in Ex. 13 the radius

of gyration of the paraboloid of the preceding example about a perpendicular to the axis passing through the vertex.

$$k^2 = \frac{3h^2 + b^2}{6}.$$

16. A solid homogeneous 8-inch shot consists of a cylinder two calibres in length and an ellipsoidal head one calibre in length; the rifling gives it a rotation about its axis of one turn in 25 feet. What ratio does the energy of rotation bear to that of translation? 1 : 300.

17. A drum whose diameter is 6 feet, and whose moment of inertia is that of 40 pounds at a distance of 10 feet from the axis, is employed to wind up a load of 500 pounds from a vertical shaft. It is rotating at the rate of 120 turns a minute when the steam is shut off. How far should the load be from the shaft's mouth that the kinetic energy of the load and drum may just suffice to carry the load to the surface? 41.9 ft.

18. The ogival head of a projectile is formed by the revolution of a semi-parabola about the ordinate $b$, so that the height $h$ is the radius of the bore or one-half the calibre $d$. Determine $k^2$ for the axis of revolution.

$$k^2 = \frac{2}{21}a^2.$$

19. Show that the moment of inertia of a uniform right prism of any cross-section about an axis in the plane of any right section is equal to the moment of inertia, about the same axis, which it would have if its mass were concentrated into the section as a lamina, increased by that which it would have if its mass were concentrated into its length as a rod passing through the axis.

## XXIII.

### Relations between Moments of Inertia about Different Axes.

459. The Statical Moment of a body with respect to a given plane is defined in Art. 178 as $\Sigma mp$, where $p$ is the distance of the particle $m$ from the plane. The values of three such statical moments, for example, with respect to three coordinate planes of

reference, serve to determine the value of the statical moment with reference to any given plane; for they determine the centre of inertia, and the statical moment is $Mp$, where $p$ is the perpendicular distance of the centre of inertia from the given plane.

In the case of moments of inertia about different axes, the relations are not so simple; but we shall find that, supposing the centre of inertia, or Centroid, already found, relations exist by virtue of which the values of the moment of inertia about three particular axes will serve to determine that with respect to any given axis.

## Moments of Inertia about Parallel Axes.

460. The first of these relations is that which exists between the moments of inertia of a body about parallel axes, one of which passes through the centre of inertia. Let Fig. 108 represent a section of the body made by a plane passing through the centre of inertia $O$, and perpendicular to the axes, one of which pierces the plane of the diagram at $O$, and the other at $A$, at a distance $OA = h$. Assume rectangular coordinate axes, $OA$ being the axis of $x$, and the centroidal axis of moments that of $z$. Denote by $I_0$ the moment of inertia about this axis, and by $I_1$ that about the parallel axis through $A$. Let $P$ be the projection upon the plane of $xy$ of the point at which the particle $m$ is situated; then $PO$ is equal to the distance of the particle from the centroidal axis, and $PA$ is equal to its distance from the parallel axis through $A$.

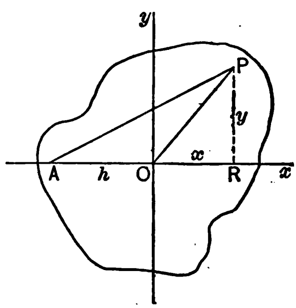

FIG. 108.

From the figure we have

$$AP^2 = y^2 + (x + h)^2 = y^2 + x^2 + 2hx + h^2;$$

$$OP^2 = y^2 + x^2.$$

Hence

$$I_1 = \Sigma mAP^2 = \Sigma mOP^2 + 2h\Sigma mx + h^2\Sigma m$$
$$= I_0 + 2h\Sigma mx + h^2\Sigma m.$$

Now $\Sigma mx = 0$, because it is the statical moment of the body with respect to the plane of $yz$ which passes through the centroid. Hence the equation reduces to

$$I_1 = I_0 + h^2M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $M$ is the total mass of the body.

Introducing the radii of gyration this equation becomes

$$k_1^2 = k_0^2 + h^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It follows that *for all parallel axes the moment of inertia (and radius of gyration) is lesat when the axis passes through the centre of inertia.*

**461.** If the moment of inertia about all axes through the centre of inertia is known, this theorem determines the moment about every axis. For example, we found in Art. 452 that the moment of inertia of a sphere about a diameter was $\frac{2}{5}a^2V$. Putting $h = a$, we have therefore for the moment of inertia about a tangent $I = \frac{7}{5}a^2V$.

**462.** In many cases, the moment of inertia about an axis not passing through the centroid is more easily found by integration than that for the centroidal axis. For example, we readily find, for the triangle about an axis through its vertex and parallel to its base, $k^2 = \frac{1}{2}h^2$, where $h$ is the altitude. Now the distance from the vertex to the centre of inertia is $\frac{2}{3}h$. Hence, by the theorem, we have, for the centroidal radius of gyration,

$$k_0^2 = \frac{1}{2}h^2 - \frac{4}{9}h^2 = \frac{1}{18}h^2.$$

Again, to find the radius of gyration about the base, that is to pass to the distance $\frac{1}{3}h$ *from* the centre of inertia, we have

$$k^2 = k_0^2 + \frac{1}{9}h^2 = (\tfrac{1}{18} + \tfrac{1}{9})h^2 = \frac{1}{6}h^2.$$

## Application to the Moment of the Element.

**463.** The application of the theorem of Art. 460 to the element of moment is often useful in enabling us to express a moment of inertia as a simple integral. For example, let it be required to find the moment of inertia of the cone represented in Fig. 109 about the axis of $z$, that is to say a perpendicular through the vertex to the geometrical axis. The only convenient element of volume for simple integration, in this case, is the circular section perpendicular to the axis of $x$. Denoting the height of the cone by $h$ and the radius of the base by $b$, the radius of this element is

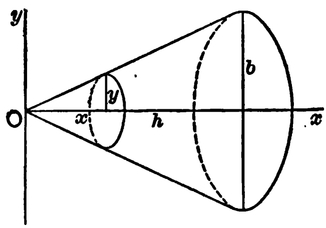


FIG. 109.

$$y = \frac{b}{h}x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The element of volume is then $\pi y^2 dx$. To find its squared radius of gyration about the axis of $z$, we notice that this axis is at a distance $x$ from the parallel diameter of the element which is a centroidal axis. The squared radius of gyration about the latter is, by Art. 448, $\frac{1}{4}y^2$; hence, by the theorem of parallel axes, that about the axis of $z$ is

$$x^2 + \tfrac{1}{4}y^2.$$

It follows that the moment of inertia of the element is

$$dI = \pi y^2(x^2 + \tfrac{1}{4}y^2)dx.$$

Substituting the value of $y$ in equation (1), we have

$$dI = \frac{\pi b^2}{h}\left(1 + \frac{b^2}{4h^2}\right)x^4 dx.$$

Therefore

$$I = \frac{\pi b^2(4h^2 + b^2)}{4h^4}\int_0^h x^4 dx = \frac{\pi b^2 h(4h^2 + b^2)}{20};*$$

and since $V = \frac{1}{3}\pi b^2 h$, $k^2 = \frac{3}{20}(4h^2 + b^2)$.

## The Principal Axes for a Point in the Plane of a Lamina.

464. We have next to consider the relations which exist between moments of inertia about axes passing through a given point. We begin with the case of a plane lamina and axes in its plane, and shall prove that, for any given point in the lamina, there are two such axes about which the moments of inertia are respectively greater and less than that about any other axis in the plane and through the point; except in the case when the moments of inertia about all such axes are equal.

465. Let $O$, Fig. 110, be the given point, and let rectangular axes through $O$ be assumed. Let another pair of rectangular axes $Ox'$, $Oy'$ make the angle $\alpha$ with those of $x$ and $y$. Then, if $P$ be the position of a particle whose mass is $m$, we readily obtain from the figure for its distances from the new axes

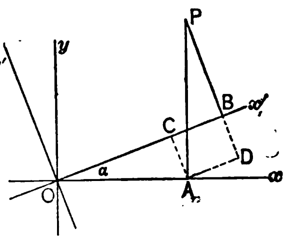


FIG. 110.

$$\left.\begin{aligned} x' &= OC + AD = x\cos\alpha + y\sin\alpha, \\ y' &= PD - AC = y\cos\alpha - x\sin\alpha. \end{aligned}\right\} \quad . \quad . \quad (1)$$

* It will be noticed that the first term of the integral, in this process, is $\Sigma mx^2$. Therefore, since $I_z = \Sigma mx^2 + \Sigma my^2$, the second term is the value of $\Sigma my^2$. (Compare the method employed in Ex. XXII, 13.) The triple integral expression for $\Sigma my^2$ would, in this case, be

$$\int_0^h\int_{-y_1}^{y_1}\int_{-z_1}^{z_1} y^2 dz\,dy\,dx,$$

in which the limits for $z$ are values in terms of $y$ and $x$ obtained from the equation of the conical surface, and those for $y$ are taken from equation (1) above.

From these equations, we have

$$x'^2 = x^2 \cos^2\alpha + 2xy \sin\alpha \cos\alpha + y^2 \sin^2\alpha,$$
$$y'^2 = y^2 \cos^2\alpha - 2xy \sin\alpha \cos\alpha + x^2 \sin^2\alpha,$$
$$x'y' = xy(\cos^2\alpha - \sin^2\alpha) - (x^2 - y^2)\sin\alpha\cos\alpha.$$

Multiplying by $m$, and summing for all the particles of the body, we obtain

$$\left.\begin{aligned} \Sigma mx'^2 &= \Sigma mx^2 . \cos^2\alpha + \Sigma mxy . \sin 2\alpha + \Sigma my^2 . \sin^2\alpha, \\ \Sigma my'^2 &= \Sigma my^2 . \cos^2\alpha - \Sigma mxy . \sin 2\alpha + \Sigma mx^2 . \sin^2\alpha, \end{aligned}\right\} \quad (2)$$

$$\Sigma mx'y' = \Sigma mxy . \cos 2\alpha - \tfrac{1}{2}\Sigma m(x^2 - y^2) . \sin 2\alpha. \quad . \quad . \quad . \quad (3)$$

Now, supposing $\Sigma mx^2$, $\Sigma my^2$ and $\Sigma mxy$ to have been found, $\alpha$ may be so taken that $\Sigma mx'y' = 0$; for, in equation (3), this gives

$$\tan 2\alpha = \frac{2\Sigma mxy}{\Sigma mx^2 - \Sigma my^2}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which is always possible, since the tangent of an angle may have any value, positive or negative. If $\alpha_0$ is a value which satisfies the equation, $\alpha_0 + 90°$ also satisfies the equation, but this change in the value of $\alpha$ only interchanges the new axes of $x$ and $y$. There is therefore, in general, but one pair of rectangular axes for which $\Sigma mxy = 0$.

466. The axes thus determined are called the axes of principal moment, or *principal axes* of the lamina for the point $O$. Suppose now that $Ox$ and $Oy$ in Fig. 110 are the principal axes, so that $\Sigma mxy = 0$; and put $I_x$ for $\Sigma my^2$, $I_y$ for $\Sigma mx^2$, as in Art. 447. Then, putting $I_\alpha$ for $\Sigma my'^2$, the moment of inertia about $Ox'$, which makes the angle $\alpha$ with the principal axis $Ox$, the second of equations (2) gives

$$I_\alpha = I_x \cos^2\alpha + I_y \sin^2\alpha, \quad . \quad . \quad . \quad . \quad (5)$$

reducing to $I_x$ when $\alpha = 0$, and to $I_y$ when $\alpha = 90°$.

If $I_x > I_y$, $I_\alpha$ is less than $I_x$ and greater than $I_y$, so that $I_x$ is

its maximum and $I_y$ its minimum value. But if $I_x = I_y$, $I_\alpha$ is constant; that is to say the moment of inertia is the same for all axes in the plane of the lamina passing through the given point.

467. As an example, let us find the principal axes of the right triangle $OAB$, Fig. 111, for the right angle $O$. Taking $OA = a$ and $OB = b$ for axes of $x$ and $y$ respectively, we have, as in Art. 462,

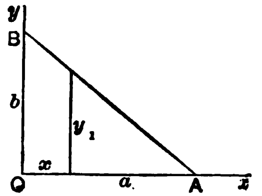

FIG. 111.

$$\Sigma mx^2 = \tfrac{1}{6}a^2 . M, \quad \Sigma my^2 = \tfrac{1}{6}\, b^2 . M.$$

For $\Sigma mxy$, we have (for unit density)

$$\Sigma mxy = \int_0^a \int_0^{y_1} xy\, dy\, dx = \tfrac{1}{2}\int_0^a xy_1^2\, dx,$$

in which, from the figure, the upper limit for $y$ is

$$y_1 = \frac{b}{a}(a - x).$$

Hence

$$\Sigma mxy = \frac{b^2}{2a^2}\int_0^a x(a-x)^2 dx = \frac{b^2}{2a^2}\int_0^a (a^2x - 2ax^2 + x^3)dx$$

$$= \frac{b^2a^2}{2}\left(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}\right) = \frac{ab}{12}M.$$

Substituting in equation (4), Art. 465,

$$\tan 2\alpha = \frac{ab}{a^2 - b^2}.$$

The principal axis of least moment here corresponds to the value $\alpha_0$ in the first quadrant; its direction lies between the medial line through $O$ and the greatest side and admits of an easy graphical construction.

### The Momental Ellipse of a Lamina for a Given Point.

468. Let $\rho$ be a length such that $\dfrac{1}{\rho^2}$ is proportional to $I_\alpha$, so that $\rho$ represents *the reciprocal of the radius of gyration*, and let $\alpha$

and $b$ be its values for $\alpha = 0$ and $\alpha = 90°$, corresponding to $I_x$ and $I_y$. Equation (5) of Art. 466 then gives

$$\frac{1}{\rho^2} = \frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2}.$$

Let $\rho$ be laid off from $O$ on the axis to which it belongs, which makes the angle $\alpha$ with the axis of $x$; so that $\rho$ and $\alpha$ are the polar coordinates of a point which, as $\alpha$ varies, describes the curve of which the above is the polar equation.

Multiplying by $\rho^2$,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the rectangular equation of this curve, which is therefore an ellipse.

Thus the radius of gyration of the lamina about an axis in its plane passing through $O$ in any direction is represented by the reciprocal of the radius vector in that direction of this ellipse, which is called *the momental ellipse* * of the lamina with respect to the point $O$.

### Principal Axes of a Lamina at the Centre of Inertia.

469. The most important principal axes of a lamina are the centroidal ones. When the lamina is symmetrical with respect to each of two rectangular axes it is easy to see that, taking them as coordinate axes, $\Sigma mxy = 0$, and therefore these axes are the principal axes. Thus, the axes of an ellipse are principal axes; the lines bisecting opposite pairs of sides of a rectangle are principal axes; the diagonals of a rhombus are principal axes.

If the principal moments are equal, the momental ellipse be-

* If we had made $\rho$ *directly* proportional to the radius of gyration, we should have obtained the curve inverse to the ellipse with respect to its centre. The reciprocal is taken because it leads to a simpler and more familiar curve.

comes a circle and the moments are equal about all centroidal axes. This is the case with the square, as we have already seen in Art. 449. Again, if three moments of inertia about axes through $O$ (whether $O$ is or is not the centroid) are equal, the momental ellipse becomes a circle. For example, this is the case at the centre of any regular polygon.

The theorem of parallel axes shows that if the momental ellipse is a circle for the centroid, it is not a circle for any other point. But if it is not a circle for the centroid, two points can be found for which it is a circle.

## The Moments of Inertia of a Solid for Axes passing through a Given Point.

**470.** In discussing the moments of inertia of a solid about axes which pass through a given point $O$, we shall at first suppose the plane of $xy$ to be a plane passing through $O$ taken at random.

The value of $z$ for any particle $m$ of the solid will not affect the values of the quantities $\Sigma mx^2$, $\Sigma my^2$, and $\Sigma mxy$. Hence we can show, exactly as in Art. 465, that new axes of $x'$ and $y'$ in the plane can be found such that $\Sigma mx'y' = 0$.

Now taking these new axes for those of $xy$, so that $\Sigma mxy = 0$, we shall have, as before, from the second of equations (2), Art. 465, when the axes are turned through any angle $\alpha$,

$$\Sigma my'^2 = \Sigma my^2 \, . \cos^2 \alpha + \Sigma mx^2 \, . \sin^2 \alpha. \quad . \quad . \quad . \quad (1)$$

But the terms of this equation are not now moments of inertia. In fact, for the solid,

$$I_x = \Sigma my^2 + \Sigma mz^2, \quad \text{and} \quad I_y = \Sigma mx^2 + \Sigma mz^2;$$

and, if $I_\alpha$ denotes the moment of inertia about the axis of $x'$ which is in the plane of $xy$ and makes the angle $\alpha$ with the axis of $x$,

$$I_\alpha = \Sigma my'^2 + \Sigma mz^2,$$

Hence, adding to equation (1) the identity

$$\Sigma mz^2 = \Sigma mz^2 \,.\, \cos^2\alpha + \Sigma mz^2 \,.\, \sin^2\alpha,$$

we have

$$I_\alpha = I_x \cos^2\alpha + I_y \sin^2\alpha \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the same relation for the solid as that found in Art. 466 for the lamina.

## The Momental Ellipsoid.

**471.** It follows that, for any plane passing through a given point, we have a *momental ellipse*, as in Art. 310, of which the radius-vector drawn from the centre in any direction is the reciprocal of the radius of gyration of the body about the corresponding axis.

Consider now the locus in space of the extremities of radii-vectores laid off in the same way for all axes passing through the given point. This locus is a surface of which we have just seen that the section by the plane through $O$ is an ellipse. Since this is true for *any* plane passing through $O$, the surface is such that *all* its plane sections through the point $O$ are ellipses. The surface is therefore that of an ellipsoid. This ellipsoid is called *the momental ellipsoid* of the solid with respect to the point $O$, which is its centre.

**472.** The principal axes of the momental ellipsoid are called *the principal axes of moment* of the solid for the given point, and the moments about them are the principal moments of inertia. If $a$, $b$ and $c$ are the reciprocals of the principal radii of gyration, the equation of the momental ellipsoid referred to the principal axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The greatest and the least moment of inertia for axes passing through the given point $O$ correspond respectively to the least and the greatest axis of this ellipsoid.

Now let $\rho$ be the length and $\alpha$, $\beta$, $\gamma$ the direction-angles of the radius-vector of the ellipsoid, so that

$$x = \rho \cos \alpha, \qquad y = \rho \cos \beta, \qquad z = \rho \cos \gamma;$$

then the equation of the ellipsoid may be written in the polar form,

$$\frac{1}{\rho^2} = \frac{\cos^2 \alpha}{a^2} + \frac{\cos^2 \beta}{b^2} + \frac{\cos^2 \gamma}{c^2}. \quad . \quad . \quad . \quad . \quad (2)$$

Denoting the moment of inertia about the axis whose direction-angles are $\alpha$, $\beta$, $\gamma$ by $I_{\alpha, \beta, \gamma}$, we have, on multiplying equation (2) by the mass $M$,

$$I_{\alpha, \beta, \gamma} = I_x \cos^2 \alpha + I_y \cos^2 \beta + I_z \cos^2 \gamma. \quad . \quad . \quad (3)$$

By means of this theorem and that expressed by equation (1), Art. 460, we can find the moment of inertia of a body about any axis, when we know the principal axes and moments for the centre of inertia.

## The Principal Axes of Symmetrical Bodies.

**473.** The principal axes of the solid *in a plane*, for which $\Sigma mxy = 0$, as found in Art. 470, are of course the axes of the ellipse in which the given plane cuts the ellipsoid. Hence, when the principal moments of the solid are all unequal, it follows from the nature of the ellipsoid that the principal planes are the only set of rectangular coordinate planes for which we have at once

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0.*$$

* By using the general equations of transformation for passing to a new set of rectangular coordinate planes with the same origin, we might have obtained expressions for $\Sigma mx'y'$, $\Sigma my'z'$ and $\Sigma mz'x'$; and then, by equating these to zero, we might have determined the position of the principal axes in terms of $\Sigma mx^2$, $\Sigma mxy$, etc., supposed to have been calculated for the assumed coordinate planes. In the process given in the text, we have confined ourselves to the proof of the existence of the principal axes and the expression of the moment of inertia about any axis in terms of the principal moments.

But if two of these equations are true, the axis in which the corresponding planes intersect is a principal axis. Thus, if $\Sigma mxz = 0$ and $\Sigma myz = 0$, the axis of $z$ is an axis of each of the ellipses in which the planes of $xz$ and of $yz$ cut the momental ellipsoid. It follows that a plane tangent to the ellipsoid at the point where the axis of $z$ cuts the surface is parallel to the plane of $xy$; hence the axis of $z$ is an axis of the ellipsoid. The other principal axes are now principal axes for the plane of $xy$, and can therefore be determined by the method illustrated in Art. 467.

**474.** The position of one or more of the principal axes of a solid is sometimes obvious from considerations of symmetry. For example, suppose the body to be homogeneous and symmetrical to a given plane, so that, taking this as the plane of $xy$, to any particle situated at a point $(x, y, z)$ there corresponds an equal particle at $(x, y, -z)$; then it is evident that, no matter where the origin and axis of $x$ be taken in the plane, we shall have $\Sigma mxz = 0$. In like manner we have $\Sigma myz = 0$. Therefore, at every point of the plane of symmetry, the line perpendicular to it is a principal axis of the body for that point. Since the centre of inertia is in the plane of symmetry, we can therefore readily find the principal centroidal axes. In accordance with this principle, a plate of uniform thickness, or any body in the form of a right prism, has at any point of its central plane the line perpendicular to it for a principal axis of inertia. It obviously follows, from Art. 447, that, for a thin plate, this axis is the axis of *greatest* moment.

**475.** If there be two planes of symmetry, we shall thus have, for any point of their line of intersection, the position of two principal axes; and, these being in the plane perpendicular to the line of intersection, that line will itself be the third principal axis. For instance, a right pyramid whose base is a rectangle has two planes of symmetry, each passing through the geometrical axis and the middle points of a pair of opposite sides of the base. Therefore, for any point of the geometrical axis, this axis itself and the lines joining the middle points of the right section of the pyramid are the axes of principal moment of inertia.

476. The two planes of symmetry are usually at right angles, as in the illustration just given, and the corresponding principal moments will generally be unequal. But if the planes cut obliquely, we have the case in which each of two oblique axes fulfils the condition for a principal axis, and therefore all the axes in the plane give equal moments of inertia (see Art. 469). An instance is afforded by a right pyramid having an equilateral triangle for its base. The moments of inertia for all axes passing through a point of the geometrical axis and in a plane parallel to the base are equal. In such a case, the momental ellipsoid becomes a spheroid. In like manner, the moments of inertia for all axes passing through the centre of a regular tetrahedron, or of any regular solid, can be shown to be equal, the momental ellipsoid for that point becoming in these cases a sphere.

## The Equimomental Ellipsoid.

477. For any given rigid body, there may be found a homogeneous ellipsoid having the same mass and the same principal moments at the centre of inertia as the given body, and therefore, from what precedes, the same moment of inertia and the same radius of gyration as the given body for every axis. This ellipsoid is called *the equimomental ellipsoid* of the body.

Let the centre of inertia of the given body be taken as the origin, and the principal axes as coordinate axes, and let $a$, $b$, $c$ be the semi-axes of an ellipsoid lying respectively in the axes of $x$, $y$ and $z$. The principal axes of inertia of this ellipsoid at the origin, which is its centre of inertia, are, by Art. 475, the coordinate axes, for which the moments of inertia were found in Art. 452 to be

$$k_z^2 = \tfrac{1}{5}(a^2 + b^2),$$
$$k_x^2 = \tfrac{1}{5}(b^2 + c^2),$$
$$k_y^2 = \tfrac{1}{5}(c^2 + a^2).$$

Now, if in these equations $k_x$, $k_y$, $k_z$ are the principal radii of

gyration of the given body, we have, by solving for $a$, $b$ and $c$, the semi-axes of the body's equimomental ellipsoid; namely,

$$a^2 = \tfrac{5}{2}(k_y^2 + k_z^2 - k_x^2),$$
$$b^2 = \tfrac{5}{2}(k_z^2 + k_x^2 - k_y^2),$$
$$c^2 = \tfrac{5}{2}(k_x^2 + k_y^2 - k_z^2).$$

This ellipsoid, which, as stated above, must also have the same mass as the given body, may be substituted for that body in any question involving either the inertia of rotation or that of translation.

## The Compound Pendulum.

478. A heavy body of any form free to turn upon a horizontal axis not passing through its centre of gravity is called a *compound pendulum*, in distinction from the simple pendulum, in which the mass is regarded as concentrated into a single particle.

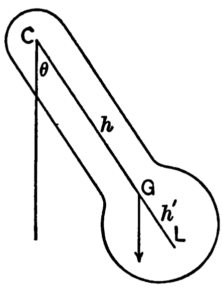

FIG. 112.

Let $G$, Fig. 112, be the centre of gravity, and $C$ the point where the axis is cut by a vertical plane perpendicular to it, passing through $G$. This point is called *the point of suspension*. The forces acting upon the body are its weight, acting vertically downward at $G$, and the resistance of the axis. Since rotation about the axis is the only motion possible, we obtain the single equation of motion required, by taking moments about $C$. Denoting $CG$ by $h$, the angle it makes with the vertical by $\theta$, and the radius of gyration for the given axis by $k$, equation (1), Art. 434, gives

$$-Mgh \sin\theta = Mk^2\frac{d^2\theta}{dt^2},$$

or, putting

$$l = \frac{k^2}{h}, \qquad (1)$$

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

This is identical with the equation of motion of a simple pendulum, Art. 374, if its length is $l$, since this makes $s = l\theta$. Hence the motion is the same as that of a simple pendulum of length $l$.

Denoting the radius of gyration about a parallel to the given axis through $G$ by $k_0$, equation (2), Art. 460, gives

$$k^2 = k_0^2 + h^2;$$

hence, putting $h'$ for $\frac{k_0^2}{h}$, so that

$$k_0^2 = hh', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

equation (1) gives

$$l = \frac{k_0^2}{h} + h = h + h'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It follows that, measuring from $C$, Fig. 110, the distance $CL = l$, which is *the length of the equivalent simple pendulum*, the point $L$ will lie on the other side of $G$ at the distance $GL = h'$. The point $L$ is sometimes called *the centre of oscillation*.

479. Equation (2) shows that $h$ and $h'$ may be interchanged, the value of $l$ being by equation (3) unchanged. Hence the remarkable result that if the body be suspended from the centre of oscillation the time of vibration remains unchanged.*

In the ordinary pendulum in which most of the mass is contained in a small bob, $k_0$ is small relatively to $h$, and therefore the centre of oscillation is but a short distance below the centre of gravity. But, making $h$ small relatively to $k_0$, $h'$ can be made as large as we please, and the centre of oscillation placed far below and outside of the body. Thus a body of limited size can be so mounted as to be the equivalent of a very long simple pen-

* This principle is used in determining experimentally the exact position of the centre of oscillation of the pendulum which beats seconds, and thence the value of $L$. This is done in "pendulum experiments" to determine the *absolute* value of $g$, by the formula $g = \pi^2 L$, Art. 382. The experiments mentioned in Art. 385 are only for *variations* in the value of $g$.

dulum. Advantage is taken of this principle in the pendulum of *the metronome*, in which the centre of gravity is adjustable, so that $h$ may be shortened, and the length $l$ (and consequently the time of vibration) increased at pleasure.

The observed time of vibration of a body mounted as a pendulum is often used to determine $k_0$, $h$ being measured, and $h'$ derived from the calculated value of $l$.

## Foucault's Pendulum Experiment.

**480.** A body at rest relatively to the earth, partakes of its rotary motion. Thus, if a body is mounted on an axis through its centre of gravity, parallel to that of the earth, it may be regarded as rotating about that axis at the angular rate of 360° a day, or 15° an hour. Even if the axis were perfectly smooth, the body would continue to rotate at this rate, and thus have no rotation relatively to the earth. But, if an ideal body without mass could be thus mounted, in such a way as not to share the rotation of the earth, it would have a rotation relatively to the earth exactly equal and opposite to the real rotation of the earth. This apparent rotation would then afford an experimental proof of the rotation of the earth.

Such an ideal body, mounted upon a vertical axis, is furnished by the plane of vibration of a pendulum so suspended as to be free to vibrate in any vertical plane. If this experiment, which was devised by Foucault, were performed at the pole of the earth, the rate of the apparent rotation of the plane of vibration would be 15° an hour.

**481.** At any other place, the rotation thus put in evidence will be only a resolved part of the earth's rotation. To find its amount, let $\lambda$ be the latitude, and suppose the line tangent to the meridian to meet the earth's axis produced in $C$: this line will describe the surface of a cone with vertex at $C$ touching the earth in the parallel of latitude. The tangent line may at any instant be regarded as rotating about $C$, and it is readily seen that its angular rate is to that of the earth inversely as the length of

the tangent is to the radius of the parallel of latitude. Thus, if $\omega$ is the angular rate of the earth, $\omega \sin \lambda$ is *the rate of rotation of the plane of the horizon* about a vertical axis, and this is the apparent rate of the plane of vibration exhibited by the experiment.*

## Pressure on the Axis of a Uniformly Rotating Lamina.

482. When a body mounted upon a fixed axis is at rest under the action of external forces, these forces are subject to one condition of equilibrium, namely, that their resultant moment about the axis of rotation shall vanish. As mentioned in Art. 242, the other five conditions of equilibrium serve to determine the reactions of the fixed axis. If this axis is, as usual, supported at two points, the pressures resisted by the supports may be reduced to three—one in the direction of the axis, and one at each support in some direction perpendicular to the axis. Since each of those last mentioned involve two unknown quantities, we have thus five quantities in all to be determined by the five conditions.

If now the body be in rotation, the rate will remain uniform because there is no moment about the axis. The inertia of any particle $m$ at the distance $r$ from the axis will now, as stated in Art. 432, consist solely of its centrifugal force, $mr\omega^2$, which acts in a line passing through the axis. We have now to consider the resultant of these centrifugal forces for all the particles of a body, and the additional pressure upon the axis thus produced.

We begin with the case of a lamina rotating about an axis perpendicular to its plane.

483. Take any rectangular axes in the plane of the lamina passing through the point in which it is pierced by the axis of rotation, and fixed with reference to the substance of the lamina.

* The experiment must be executed with great care to prevent lateral motion. Otherwise, the rotation of the longer axis of the orbit mentioned in the foot-note, p. 317, will completely disguise the motion to be exhibited.

Let $x$ and $y$ be the rectangular, and $r$ and $\theta$ the polar, coordinates of the particle $m$, as referred to these axes. Then, $\omega$ being the angular velocity, the centrifugal force $m\omega^2 r$ acts at the origin, or centre of rotation, in the direction making the angle $\theta$ with the axis of $x$. Hence the resolved part of this force along the axis of $x$ is

$$m\omega^2 r \cos\theta = m\omega^2 x. \quad \ldots \ldots \quad (1)$$

It follows that the resolved part, along this axis, of the resultant of the whole system of centrifugal forces is

$$X = \omega^2 \Sigma mx. \quad \ldots \ldots \quad (2)$$

Now $\Sigma mx$ is the statical moment of the whole mass, $M = \Sigma m$, with respect to the plane of $yz$; so that $\Sigma mx = M\bar{x}$, where $\bar{x}$ is the abscissa of the centre of inertia. Thus the resolved part of the resultant of the centrifugal forces has the value

$$X = \omega^2 M\bar{x},$$

which, in accordance with expression (1), is the same that it would have if all the mass were concentrated at the centre of inertia. In like manner, the resolved part of the resultant along the axis of $y$ is $\omega^2 M\bar{y}$, which is the same as if all the mass were concentrated at the centre of inertia. Hence the resultant centrifugal force is

$$R = \omega^2 M\bar{r}, \quad \ldots \ldots \quad (3)$$

where $\bar{r}$ is the distance of the centre of inertia from the centre of rotation, and this force acts in a line directed toward the centre of inertia. In other words, *for a lamina rotating about an axis perpendicular to its plane, the resultant centrifugal force is the same as if the whole mass were concentrated at the centre of inertia.*

484. This centrifugal force $M\omega^2\bar{r}$ acts upon the axis, at the point where it pierces the lamina, in a line which rotates with the body. If the axis is supported at two points or pivots, say one on each side of the lamina, the pressures upon these supports will

be parallel components of the centrifugal force (3), acting in lines which, in like manner, rotate with the body, and their magnitudes will be found as in Art. 87.

In particular, *if the axis of rotation passes through the centre of inertia of the lamina, and is perpendicular to its plane, there will be no pressure upon the supports to the axis resulting from the rotation.*

## Pressure on the Axis of a Uniformly Rotating Solid.

**485.** Passing now to the general case, let the rigid body of any form be referred to any three rectangular axes, of which that of $z$ is the axis of rotation. Suppose the body separated into laminæ by planes perpendicular to the axis of rotation, each lamina being characterized by a particular value of $z$. The centrifugal force due to a particular lamina acts upon the axis at the point (o, o, $z$), and has no component in the direction of the axis of $z$. Its components in the direction of the other axes are, by Art. 483,

$$X = \omega^2 \Sigma_{x,y} mx, \qquad Y = \omega \Sigma_{x,y} my,$$

where $\Sigma_{x,y}$ indicates summation extended to particles having all values of $x$ and $y$, but only the given particular value of $z$. Now, substituting in the equations of Art. 232, we have for the moments of this force about the axes of $x$ and $y$

$$L = -zY = -\omega^2 z \Sigma_{x,y} my, \qquad M = zX = \omega^2 z \Sigma_{x,y} mx,$$

and $N = 0$.

**486.** The six elements (Art. 234) of the system consisting of the centrifugal forces of all the laminæ are found by summing the expressions above for all values of $z$. Thus they are

$$\Sigma X = \omega^2 \Sigma mx, \qquad \Sigma Y = \omega^2 \Sigma my, \qquad \Sigma Z = 0, \quad . \quad (1)$$

$$\Sigma L = -\omega^2 \Sigma myz, \quad \Sigma M = \omega^2 \Sigma mzx, \quad \Sigma N = 0, \quad . \quad (2)$$

where the summations in the second members now extend to all the particles of the body.

It follows that the system of centrifugal forces is equivalent to a dyname $(R, K)$ in which the force $R$ is the resultant of $\Sigma X$ and $\Sigma Y$ acting at the origin, and the couple $K$ is the resultant of the couples $\Sigma L$ and $\Sigma M$. Comparing with Art. 483, we see that, he value of $R$, vectorially considered, has the same expression as in the case of the lamina, namely,

$$R = \omega^2 M\bar{r},$$

so that it has the same value and the same direction as if the whole mass were concentrated at the centre of inertia. But $R$ is now regarded as acting at the origin, which may be any point upon the axis, and the value of $K$ depends, as explained in Art. 226, upon the position chosen for the origin.

487. The force $R$ and the axis of the couple $K$ both lie in the plane of $xy$; but they will not generally be at right angles; so that *the system cannot generally be reduced to a single force.* If the axis is supported at two points, as in Art. 484, the pressures upon the supports will now consist not only of a component of $R$ at each point parallel to the direction of $R$; but, in addition to these, of two equal and opposite forces, one at each point in a direction perpendicular to the axis of $K$. If $a$ is the distance between the supports, the value of either of these forces is $Q$, where $K = aQ$.

## Condition under which the Centrifgual System is Equivalent to a Single Force.

488. Substituting the values found in Art. 486, the condition under which the system is reducible to a single force (see Art. 236) becomes

$$\Sigma mx \,.\, \Sigma myz - \Sigma my \,.\, \Sigma mxz = 0. \quad . \quad . \quad . \quad . \quad (1)$$

This is satisfied when $\Sigma myz = 0$ and $\Sigma mxz = 0$, which, as we have seen in Art. 473, is the condition that the axis of $z$ shall be a principal axis for the origin. In this case $K = 0$, and the sys-

tem of centrifugal forces reduces to the force $R$ acting at the origin.

The condition is also fulfilled by any axis which lies in a plane of symmetry. For suppose this plane to be taken as the plane of $xz$, and let $b$ denote a special value of $y$; then, by hypothesis, the laminæ parallel to the plane of $xz$ corresponding to $y = b$ and to $y = -b$ are precisely alike. Now, for the first of these laminæ, the first member of equation (1) becomes

$$b\Sigma mx \,.\, \Sigma mz - b\Sigma m \,.\, \Sigma mxz.$$

For the lamina $y = -b$, this expression changes sign; since, by the identity of the laminæ $\Sigma m$, $\Sigma mx$, $\Sigma mz$ and $\Sigma mxz$ are unchanged. Hence, for the two laminæ taken together, equation (1) is satisfied; and, since the body consists of such pairs of laminæ, it is satisfied for the whole body.

It follows that, for an axis in a plane of symmetry, the system of centrifugal forces is equivalent to a single force equal to the centrifugal force of the whole mass supposed concentrated at the centre of inertia; but it must be remembered that this force does *not* generally act at that point.

489. In the case of a lamina, the foregoing applies to any axis in the plane of the lamina. As an illustration, suppose the triangular lamina, Fig. 111, Art. 467, to be rotating about the side $OB$, which (to agree with the notation of the preceding articles) we now take as the axis of $z$. The value of $R$ is the centrifugal force of the whole mass $M$ rotating at the distance of its centre of gravity from the axis, which is $\frac{1}{3}a$. Thus $R = \frac{1}{3}\omega^2 Ma$. Since in this case $\Sigma L = 0$, the couple $K$ is the same as $\Sigma M$ of Art. 486, and its plane is the plane of $xz$, which is the plane of the lamina. Hence it can be combined with $R$ acting at the origin, as in Art. 101, the resultant being $R$ acting at the point $(0, z_1)$, where

$$Rz_1 = K = \omega^2 \Sigma mzx.$$

Now, as found in Art. 467, we have, in this case, $\Sigma mzx = \frac{1}{12}abM$, hence $z_1 = \frac{1}{4}b$; that is, the resulting centrifugal force of a homo-

geneous right triangle rotating about a side acts upon the axis at a distance from the right angle equal to one-fourth of that side.*

## Rotation about a Centroidal Axis.

490. If, in the equations of Art. 486, $\Sigma mx = 0$ and $\Sigma my = 0$, we have $R = 0$,—that is to say, if the axis of rotation passes through the centroid, the resultant of the centrifugal forces is the couple $K$. The pressures produced upon two fixed supports to the axis at the distance $a$ apart are, in this case, simply equal and opposite parallel forces acting in lines perpendicular to the axis of $K$, and of magnitude $Q$, where $aQ = K$.

If, in addition to these conditions, we have $\Sigma myz = 0$ and $\Sigma mxz = 0$, the couple $K$ also vanishes, and the system of centrifugal forces is in complete equilibrium. In this case, the axis of $z$ is, by Art. 473, a principal axis for the origin. It is readily shown that it is also a principal axis for the centroid, so that the three centroidal principal axes are the only ones about which the centrifugal forces are in equilibrium, except in the special cases mentioned in Art. 476, where two or all three of the principal moments of inertia are equal, that is, when the centroidal momental ellipsoid becomes a spheroid or a sphere.†

491. If there are no external forces acting upon a body rotating about a centroidal principal axis, there will be no pressure what-

* It is beyond the scope of the present volume to go further into the discussion of the principal axes of a body. It may, however, be here stated that every axis about which the centrifugal forces reduce to a single force is a principal axis of inertia of the body, but in general it is a principal axis only for the point at which the force acts. Thus, in the illustration above, $OB$ is a principal axis for the point $(0, \frac{1}{4}b)$, (the other two principal axes for this point being a perpendicular to the plane and a parallel to $OA$). For a principal axis passing through the centroid, there is no force $R$ to define the point for which it is a principal axis, and accordingly such an axis is a principal axis for every one of its points.

† In the first of these cases the body is said to have kinetic symmetry with respect to an axis, and in the second to have complete kinetic symmetry.

ever upon the axis and it may remain unsupported. The body is then said to *rotate freely* about the axis. But it can be shown that it is only in the case of the centroidal principal axis of greatest moment that the rotation is in kinetic stability. Thus, in accordance with Art. 474, the rotation of a thin plate about an axis through its centre of inertia and perpendicular to its plane is stable.

## Pressure on the Axis when the Rotation is not Uniform.

**492.** When a body mounted on a fixed axis is subject to external forces which have a resultant moment about the axis, the body will not only be in rotation but will have an angular acceleration. The pressures upon the supports to the axis will now consist not only of those due to the external forces and to the centrifugal forces, but also of those due to another set of inertia forces, namely, the tangential components of the inertia of the various particles.

Referring to rectangular axes as in Art. 483, this component of the inertia of the particle $m$ is

$$mr\frac{d\omega}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and it acts in a direction at right angles to the direction of the radius vector $r$. This force acting at the particle is equivalent to a parallel force acting at the origin together with a couple. The resultant of these couples for all particles of the body is the moment of inertia which we have already discussed. The forces constitute a system of the form (1) acting at the same points as the centrifugal forces, considered in the preceding articles, of which the form is $mr\omega^2$. Each force of this system, in fact, differs from the corresponding force of the centrifugal system only in containing the common factor $\frac{d\omega}{dt}$ in place of $\omega^2$, and in acting in a

direction which is 90° *behind or in advance of its line of action according as the angular acceleration is positive or negative.*

It follows that the resulting pressures upon the supports to the axes bear these relations, as to magnitude and direction, to the pressures which have been discussed in the preceding articles.

## Plane Motion of a Rigid Body.

493. A lamina moving in its own plane, or a rigid body so moving that a certain plane section of it remains always in the same plane, is said to have *plane motion.*

The forces acting upon a body in plane motion, including those of inertia, are assumed to act in the plane ; for, if the body is constrained to have plane motion, we need only consider the resolved parts of the forces which lie in the plane. In the case of a solid, the particles actually move in the direction of lines parallel to the plane of reference. Thus the inertia forces considered are equal and parallel to the actual ones, and are the same as if the particles were all projected on the plane ; so that the body may be replaced by a lamina, which will, however, in general be one of varying density.

494. Any plane motion can be resolved into two component motions ; one being a rotation about any selected point $O$, and the other the motion of translation represented by the motion of the point $O$. We have had an illustration in the rolling wheel of Art. 41. It was there shown that, for any point of the rim, the velocity was at any instant the resultant of that due to the rotation about the centre, and the velocity of the centre itself, which was, in that case, a uniform velocity in a straight line. The same thing is true of any other point connected with the wheel.

495. The total momentum of a solid in motion is the sum of the momenta of all its particles. Now, remembering that momentum is a vector quantity, it follows from the preceding article that the total momentum, at any instant, of a solid in plane motion is the sum of that due to the motion of translation rep-

resented by the motion of $O$, and that due to the rotation about $O$ as a fixed point. The first of these parts is obviously the same as if the whole mass were concentrated at $O$.

**496.** To find the part due to the rotation, let rectangular axes passing through $O$ be assumed, and let $x, y$ be the rectangular and $r$, $\theta$ the polar coordinates of a particle $m$. Then, denoting the angular velocity of rotation by $\omega$, its linear velocity is $r\omega$, and its direction, supposing $\omega$ positive, makes the angle $\theta + 90°$ with the axis of $x$. It follows that the resolved velocity of $m$ in the direction of the axis of $x$ is $-r\omega \sin \theta$; hence, because $y = r \sin \theta$, the resolved momentum of $m$ in this direction is

$$m\frac{dx}{dt} = -my\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Summing, we have, for the total resolved momentum along the axis of $x$,

$$\Sigma m\frac{dx}{dt} = -\omega\Sigma my = -M\omega\bar{y}, \quad . \quad . \quad . \quad . \quad (2)$$

where $\bar{y}$ is the ordinate of the centre of inertia.

Comparing with equation (1), we see that this component of the total momentum is the same as that of a particle of mass $M$ situated at the centre of inertia. The same thing may be proved, in like manner, of the resolved part of the momentum along the axis of $y$. Therefore the total momentum due to rotation is the same as if, during the rotation of the body about $O$ as a fixed point, the whole mass were concentrated at the centre of inertia.

**497.** Combining the results proved in the preceding articles, we find that *the total momentum of a body in plane motion is the same as if the whole mass were concentrated at the centre of inertia.* When the centre of inertia is itself taken as the point of reference, the momentum due to the rotation vanishes.*

* That is to say, the linear momentum vanishes. The body possesses in virtue of its rotation an analogous property called angular momentum, which will be considered in the next chapter, Art. 524.

## Rotation and Translation Combined.

**498.** Let us suppose a lamina in plane motion to be acted upon by no external forces, and let us take the centre of inertia, $G$, as the point of reference. The lamina is thus regarded as rotating at any given instant about $G$, while $G$ is moving at the instant in a certain direction. Suppose now that $G$ were so constrained by proper guides that it could move only in the straight line having this direction. No pressure upon the guides will be produced by the motion of the mass regarded as concentrated at $G$, because the motion is in a straight line ; at the same time, by Art. 490, no pressure will be produced by the centrifugal forces due to the rotation, since $G$ is the centre of inertia. It follows that the constraint may be dispensed with, that is to say, *the lamina will continue to rotate uniformly about the centre of inertia while that point describes a straight line with uniform speed.*

It is obvious that the same reasoning applies to the case of a rigid body rotating about a principal axis through the centre of inertia, $G$, because the centrifugal forces are, by Art. 490, in equilibrium. In this motion, the axis of rotation remains fixed in the body and retains its direction in space, and the straight line in which $G$ moves may make any angle with it. For stability of motion it is, however, necessary that the axis shall be that of greatest moment of inertia.

**499.** If an external force act upon the body at $G$, or a system of forces such as those of gravity whose resultant acts at $G$, the rate of rotation will remain uniform and the motion of $G$ will be the same as if the entire mass were concentrated at $G$.

Now suppose a force to act in one of the principal planes, but not at the centre of inertia. By Art. 102, this force is equivalent to an equal force acting at the centre of inertia, and a couple whose moment is the moment of the force about the centre of inertia. The force will therefore produce the same motion of translation as if it acted at the centre of inertia. In addition, the couple will produce the same angular acceleration (determined by

the equation of Art. 434) which it would produce if the axis perpendicular to the plane and passing through the centre of inertia were fixed.

**500.** For example, suppose the fly-wheel in Fig. 104, p. 354, instead of having its axis fixed, were resting with its plane horizontal upon a smooth plane (so that the resistance of the plane neutralized the weight). The effect of the force $P$ will now be a linear acceleration of the motion of the centre determined by

$$P = M\frac{d^2s}{dt^2},$$

as well as an angular acceleration of rotation determined as in Art. 435 by

$$bP = a^2M\frac{d^2\theta}{dt^2}.$$

In explanation of the double effect of the force, it is to be noticed that, in this case, the force works through a greater space than before, and so produces an additional amount of kinetic energy which takes the form of energy of translation.

**501.** In the following example, two similar conditions of kinetic equilibrium serve to determine an unknown force, as well as the two accelerations which, by reason of a known relation which exists between them, constitute but one unknown quantity :

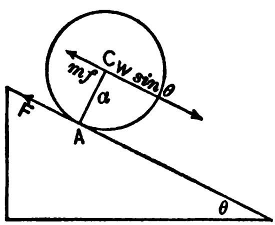


FIG. 113.

Let a cylinder whose centre of gravity is on its axis be placed with its axis horizontal upon an inclined plane rough enough to compel it to roll without slipping. Let $C$, Fig. 113 (which represents a section made by a plane perpendicular to the axis), be the centre of gravity, at which acts the weight $W = mg$, and let $A$ be the point of contact. Leaving out the component of $W$ normal to the plane and the normal resistance at $A$ (which, having the same line of action, is in equilibrium with it), the impressed

forces are the component $W \sin \theta$, acting at $C$ parallel to and down the plane, and the friction $F$ acting at $A$ up the plane. When the body is rolling down the plane, the forces of inertia which are in kinetic equilibrium with these forces are equivalent to the force $mf$, where $f$ is the linear acceleration, together with the couple which resists angular acceleration. The force $mf$ is represented in the diagram as acting at $C$. Equilibrium of the forces parallel to the plane gives

$$mg \sin \theta = F + mf. \quad \ldots \ldots \quad (1)$$

The moment of the couple to which the forces represented are equivalent is $Fa$*; hence, by equation (1), Art. 434,

$$Fa = mk^2 \frac{f}{a}, \quad \ldots \ldots \quad (2)$$

where $k$ is the radius of gyration about $C$, and the angular acceleration is $\frac{f}{a}$, because, when rolling takes place, the linear and angular velocities are connected by the relation $v = a\omega$.

Eliminating $F$ between equations (1) and (2), we find

$$ga^2 \sin \theta = (k^2 + a^2)f; \quad \ldots \ldots \quad (3)$$

whence

$$f = g \frac{a^2 \sin \theta}{a^2 + k^2},$$

and, substituting in equation (2),

$$F = W \frac{k^2 \sin \theta}{a^2 + k^2}.$$

**502.** If the cylinder is homogeneous, $k^2 = \frac{1}{2}a^2$, and we find $f = \frac{2}{3}g \sin \theta$; that is, the constant linear acceleration is two-thirds

* The forces in the diagram do not represent complete kinetic equilibrium because we have not represented in it the inertia couple which balances this couple.

of what it would be if the plane were smooth. Accordingly, when the cylinder has descended through a given vertical height, two-thirds of the potential energy expended appears in the form of energy of translation and one-third as energy of rotation. There is in this example no work done against friction, and therefore no energy lost. If we regard one-third of the work of gravity as done against the force $F$, we must also regard $F$ as doing the same work against the rotational inertia.

## EXAMPLES. XXIII.

1. Determine the radius of gyration of the paraboloid, height $h$ and radius of base $b$, about a diameter of the base.

$$k^2 = \tfrac{1}{6}(h^2 + b^2).$$

2. Find the radius of gyration of a right triangle whose sides are $a$ and $b$ about an axis perpendicular to its plane and bisecting its hypothenuse.

$$k^2 = \frac{a^2 + b^2}{12}.$$

3. Determine the radius of gyration of a thin spherical shell of radius $a$ about a tangent. $k^2 = \tfrac{5}{3}a^2$.

4. Determine the radius of gyration of a thick shell about a tangent, the exterior and interior radii being $a$ and $b$.

$$k^2 = \frac{7(a^4 + a^3b + a^2b^2) + 2(ab^3 + b^4)}{5(a^2 + ab + b^2)}.$$

5. Show that the values of $k^2$ for a homogeneous rectangular prism about its edges whose lengths are $a$, $b$ and $c$ are

$$\tfrac{1}{3}(b^2 + c^2), \qquad \tfrac{1}{3}(c^2 + a^2) \qquad \text{and} \qquad \tfrac{1}{3}(a^2 + b^2),$$

and that these are also the principal values at the centre for the prism whose sides are $2a$, $2b$ and $2c$.

6. Show that the squared radius of gyration of the second prism in Ex. 5 about its diagonal, is $k^2 = \dfrac{2(a^2b^2 + b^2c^2 + c^2a^2)}{3(a^2 + b^2 + c^2)}$, and therefore that of the prism whose edges are $a$, $b$ and $c$ about a diagonal is

$$k^2 = \frac{a^2b^2 + b^2c^2 + c^2a^2}{6(a^2 + b^2 + c^2)}.$$

7. A body consists of a hemisphere and a cone of the same base and of height equal to the radius. Determine the radius of gyration about an axis through the vertex and parallel to the common base.
$$k^2 = \frac{101}{60} a.$$

8. Find the radius of gyration of a cone about a diameter of its base, the radius being $b$ and the altitude $h$.
$$k^2 = \frac{3b^2 + 2h^2}{20}.$$

9. Show that for a point on the circumference of a circular hoop the principal axes are a perpendicular to the plane, a tangent and a diameter; and, $a$ being the radius, determine the principal moments of inertia. $2a^2M$; $\frac{3}{2}a^2M$; $\frac{1}{2}a^2M$.

10. If the centre of inertia of a lamina referred to rectangular axes is at the point $(h, k)$, and $x_0, y_0$ denote the coordinates of the point $(x, y)$, when referred to parallel axes and the centre of inertia as origin, prove that

$$\Sigma mxy = \Sigma mx_0y_0 + Mhk.$$

11. By means of the theorem of the preceding example, show, from the results in Art. 467, that the direction of the principal axes at the centre of inertia of the triangle in Fig. 111 is determined by

$$\tan 2\alpha = \frac{-ab}{a^2 - b^2},$$

and that those at the middle point of the hypothenuse are parallel to the sides. Verify the last result by considerations of symmetry.

12. Prove that a centroidal principal axis of any solid is a principal axis for every one of its points.

13. Find the moment of inertia of a cone, radius of base $b$ and height $h$, about an element or slant height.
$$\frac{3b^2(6h^2 + b^2)M}{20(b^2 + h^2)}.$$

14. A perfectly flexible cord is wrapped round a homogeneous cylinder of radius $a$. The cord is hauled in as the cylinder falls

with its axis horizontal. With what acceleration is it hauled in if the cylinder falls with the acceleration $\frac{1}{2}g$? $\frac{1}{2}g$.

15. A rod of length $b$, bent into the form of a cycloid, oscillates about a horizontal line joining its extremities. Find the length of the equivalent simple pendulum. $\frac{1}{5}b$.

16. What must be the ratio of the radius to the height of a cone in order that the centre of oscillation may be in the base when that of suspension is the vertex? Equality.

17. The height of the eaves and the ridge of a roof are $37\frac{1}{2}$ and $46\frac{1}{2}$ feet, respectively, and the slope of the roof is $30°$. Find the time in which a homogeneous sphere rolling from the ridge will strike the ground. 3 seconds.

18. What is the angle at the vertex of the isosceles triangle of given area which oscillates in the least time about an axis through its vertex and perpendicular to its plane? $90°$.

19. What is the ratio of the times of vibration of a homogeneous thin circular plate about a tangent and about a line through the point of contact perpendicular to the plane?

$\sqrt{5} : \sqrt{6}$.

20. A circular arc oscillates about an axis through its middle point and perpendicular to its plane. Show that the length of the equivalent simple pendulum is independent of the extent of the arc.

21. What is the least value of the coefficient of friction for which rolling will take place in the case of the cylinder of Fig. 113, p. 400, supposed homogeneous? $\frac{1}{3}\tan\theta$.

22. If an inclined plane is just rough enough to insure the complete rolling of a homogeneous cylinder, show that a thin hollow drum will roll and slip, the rate of slipping at any instant being one-half the linear velocity.

## CHAPTER XII.

# MOTION PRODUCED BY IMPULSIVE FORCE.

## XXIV.

### Effect of Impulsive Force.

503. A force which acts for so short a time that neither the intensity of the force nor the time of action can be directly measured is called an *impulsive force*. Such a force is, for example, called into action when a body receives a blow from a moving body, or, while moving, comes into contact with a body at rest. Let $\tau$ denote the short interval of time during which the action takes place; and let $F$ denote the intensity of the force, which may undergo variation during the interval $\tau$. The acceleration which takes place, during the interval, in the body which receives the action, is proportional to the force, and like it cannot be measured; but the whole change of velocity produced is measurable. Now we have seen, in Art. 22, that the whole change of momentum produced is the measure of the impulse. Hence, if $m$ is the mass acted upon, we have

$$\int_0^\tau Fdt = mv - mv_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $v$ and $v_0$ are the velocities before and after the impulse.

504. Accordingly, in treating of the effects of impulsive forces, our equations deal, not with the force $F$ itself, but with the impulse, which may be denoted by $(F)$. This impulse has, of course, a definite direction, which, in equation (1),

was assumed to be the direction of the velocities $v$ and $v_0$. But, by the Second Law of Motion, a velocity transverse to the direction of $(F)$ may exist without modifying this equation. Hence the equation

$$(F) = mv - mv_0$$

applies to a body having any motion, provided $v_0$ and $v$ denote the resolved velocities in the direction of $(F)$ before and after the impulse respectively.

505. The shorter we suppose the interval $\tau$ to be, in a given impulse, the greater must we suppose $F$ to be. In default of any knowledge of $\tau$, except that it is small, we are compelled to assume it so small, and $F$ so large, that the effect of any ordinary force, and the change of position due to any existing velocity, during the interval $\tau$ may be neglected. For example, when a body moving in a straight line strikes a fixed wall, it receives an impulse at the moment of contact which changes its direction; it then moves off in another straight line. We are obliged to assume that the two straight parts of the path meet at an angle, as $ABC$, Fig. 114, whereas in reality they must be connected by a very sharp curve.

We may indeed define an impulsive force as *a force which is assumed to produce a sudden change of motion*, and the impulse as *the total action of the force*; and this, in the case of a freely moving body, is measured by the momentum produced in the direction of the force.

## Impact upon a Fixed Plane.

506. When a particle comes into collision with, or *impinges upon*, a fixed plane, the resistance $R$ of the plane becomes an impulsive force which, acting for a very short time, gives to the particle an impulse in the outward direction, normal to the plane. This impulse must at least be sufficient to destroy the component of momentum in the opposite direction. Thus, if the particle $m$ moving in the line $AB$, Fig. 114, with the velocity $u$ meets the plane at $B$, the impulse it receives is in the direction $BD$ normal to the plane. Denoting the angle $ABD$, which is called *the angle*

*of incidence*, by $\alpha$, the component of momentum in the direction $DB$ is $mu \cos \alpha$. If the impulse is only sufficient to destroy this momentum, so that $(R) = mu \cos \alpha$, the particle will move with its resolved velocity $u \sin \alpha$ along the plane. In this case, that part of the kinetic energy which corresponds to the resolved velocity destroyed, namely, $\frac{1}{2}mu^2 \cos^2 \alpha$ (see Art. 323), has disappeared, and work has been done upon the body. The body may be broken or have its shape changed by this work.

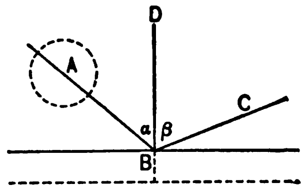


Fig. 114.

507. We may substitute for the particle considered above, moving in the line $AB$, a homogeneous sphere whose centre moves in that line, removing the plane, as indicated by the dotted line, a distance equal to the radius of the sphere, so that the centre is at $B$ when the sphere is in contact with the plane. The impulsive resistance acts along a radius of the sphere, and therefore through its centre of inertia, so that the motion is one of simple translation, and the centre of the sphere moves as a particle.

508. In general it is found that the centre moves off in a line such as $BC$ in the plane containing $AB$ and the normal. It is then said to be *reflected* from the plane, and the angle $DBC$ is called *the angle of reflection*. Denoting this angle by $\beta$, and the velocity in $BC$ by $v$, we have

$$u \sin \alpha = v \sin \beta,$$

because the resolved velocity along the plane has suffered no change. But, in this case, beside the impulse $mu \cos \alpha$ in the direction $BD$ (which has *destroyed* the original momentum in the opposite direction), the body has received a further impulse, which has *produced* the momentum $mv \cos \beta$ in the direction of the impulse. We infer, therefore, that the body has a power of regaining its form (like an elastic spring) after being compressed, so that a portion of the work done upon it by the first impulse, or *impulse of compression*, is for the instant converted into

potential energy, and that the second impulse is due to the expenditure of this potential energy and its reconversion into kinetic energy.

This property of certain bodies is called *elasticity*, and the impulsive force exerted in the second impulse is called *the force of restitution.*

509. It is obvious that the kinetic energy restored by the second impulse cannot exceed that lost in the first; therefore the second impulse cannot exceed the first. It was experimentally found by Newton that its ratio to the first impulse is independent of the magnitude of the impulse, depending only upon the material of the bodies in impact. We shall assume, therefore, that the second impulse is $e$ times the first impulse, where $e$ is a proper fraction which is called *the coefficient of restitution.*

A body for which $e = 1$ is said to be *perfectly elastic*, and one for which $e = 0$ is said to be *inelastic.*

510. In the case of impact against a fixed plane, as in Fig. 114, we shall therefore have

$$v \cos \beta = e u \cos \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this, together with

$$v \sin \beta = u \sin \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

gives, to determine the angle of reflection,

$$\tan \beta = \frac{1}{e} \tan \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and, for the velocity after impact,

$$v^2 = u^2 (\sin^2 \alpha + e^2 \cos^2 \alpha). \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The total impulse due to the resistance of the plane is now

$$(R) = mu (1 + e) \cos \alpha.$$

The kinetic energy actually lost is $\frac{1}{2}mu^2 - \frac{1}{2}mv^2$, which, by equation (4), reduces to

$$\tfrac{1}{2}mu^2 \cos^2 \alpha \,(1 - e^2).$$

This is therefore the difference between the energy lost in the first part of the impulse and that restored in the second part, or impulse of restitution.

When the body is perfectly elastic, we find $\tan \beta = \tan \alpha$, that is, the angle of reflection is equal to the angle of incidence, and $v = u$, so that there is no loss of energy. The total impulse is, in this case, double that which would be received if the body were inelastic.

**511.** If the body be subject to any continuous forces, the impulse will, in accordance with the assumption of Art. 505, produce a sudden change in the velocity and direction of the motion, and the changed values become initial values for the subsequent motion. For example, if an elastic body, describing a parabolic trajectory, strikes a fixed horizontal plane, it will rebound and describe a new parabola, the angles made with the vertical by the tangents at the point of impact corresponding to the angles of incidence and reflection above. The horizontal velocity will remain unchanged, and the vertical velocities at the point of impact will be those due to the greatest heights before and after impact. Since these vertical velocities are as $1 : e$, and the heights are as the squares of the velocities, if $h$ is the height from which a body drops, or the greatest height in the parabolic path before impact, the height to which it will rise is $h' = e^2h$.

## Direct Impact of Spheres.

**512.** In the preceding articles, we have supposed collision or impact to take place between a body of mass $m$ and a fixed or immovable body; that is to say, a body whose mass is so great that the reaction of the impulse is assumed to produce no motion in it. We now suppose impact to occur between two bodies of masses $m$ and $m'$; in this case, the impulsive force due to the im-

pact acts as a repulsive stress between them to prevent their approach, and also, in the case of elastic bodies, afterwards to cause them to separate. This stress acts in the common normal to the surfaces at the point of contact, which, when the bodies are homogeneous spheres, passes through the centre of inertia of each body, so that the motion may be treated as that of particles. We shall suppose the bodies to be such spheres, and at first that the impact is *direct*, that is, that the motion of the centres is confined to a single straight line, which will therefore be the line of action of the impulse.

Let the velocities of the masses $m$ and $m'$ before impact be $u$ and $u'$. Taking the direction in which $m$ moves as the positive one, let $u > u'$, so that, if $u'$ is positive $m$ overtakes $m'$, and if $u'$ is negative the bodies meet. In either case, the action of $m$ upon $m'$ is an impulse in the positive direction; and, by the Law of Reaction, the action of $m'$ upon $m$ will be an impulse in the negative direction, which is equal to the other in magnitude, because the time of action as well as the intensity of the impulsive force is the same for each.

513. In the first place, *let the bodies be inelastic*, then the impulse will just suffice to prevent the further approach of the bodies, but not to separate them after contact; therefore after impact they will move with a common velocity $V$. The momentum of $m$ before impact is $mu$, and after impact it is $mV$, therefore $m$'s loss of momentum is $m(u - V)$. The impulse is equal to this loss, and is also equal to the gain of momentum in the positive direction by $m'$; hence, denoting it by $(R)$, we have

$$(R) = m(u - V) = m'(V - u'); \quad . \quad . \quad . \quad (1)$$

whence, eliminating $(R)$,

$$(m + m')V = mu + m'u'. \quad . \quad . \quad . \quad . \quad (2)$$

This last equation expresses that *the whole momentum after impact is the same as that before impact.*

Solving equations (1) and (2) for $V$ and $(R)$, we have

$$V = \frac{mu + m'u'}{m + m'}, \quad \ldots \ldots \quad (3)$$

$$(R) = \frac{mm'(u - u')}{m + m'}. \quad \ldots \ldots \quad (4)$$

Thus the common velocity $V$ is the weighted mean between the velocities $u$ and $u'$, and it is to be noticed that the impulse $(R)$ is proportional to $u - u'$, that is, to *the relative velocity* of $m$ with respect to $m'$, or *the velocity of impact.* It is, in fact, the momentum of a mass equal to one-half the harmonic mean between the given masses and moving with this velocity.

**514.** Next *let the bodies be elastic*, the coefficient of restitution being $e$. Then, after they have been reduced by the first impulse to the common velocity $V$, another impulse $e(R)$ is received, acting as before in the negative direction upon $m$, and in the positive direction upon $m'$. Denote by $v$ and $v'$ the final velocities of $m$ and $m'$, then $m(V - v)$ is the momentum lost by $m$, and $m'(v' - V)$ that gained by $m'$ through this impulse. Thus

$$e(R) = m(V - v) = m'(v' - V), \quad \ldots \quad (5)$$

and these equations, solved as in the preceding article for $V$ and the impulse $e(R)$, give

$$V = \frac{mv + m'v'}{m + m'}, \quad \ldots \ldots \quad (6)$$

$$e(R) = \frac{mm'(v' - v)}{m + m'}. \quad \ldots \ldots \quad (7)$$

Comparison of equations (3) and (6) shows that

$$mv + m'v' = mu + m'u'; \quad \ldots \ldots \quad (8)$$

hence, as before, *the whole momentum is the same after impact as before impact.* Again, comparison of equations (4) and (7) gives

$$v - v' = -e(u - u'), \quad \ldots \ldots \quad (9)$$

which shows that *the relative velocity after impact is numerically $e$ times that before impact, and in the opposite direction.*

515. Equations (8) and (9), which express principles readily remembered, should be used in solving all problems involving the four velocities, the value of $e$ and the ratio of the masses, any two of which may be the unknown quantities. Since the weights $W$ and $W'$ have the same ratio as the masses, they may take the place of $m$ and $m'$ in equation (8).

A case of particular interest is that in which the weights of the spheres are equal. The equations then become

$$v + v' = u + u',$$

$$v - v' = -eu + eu'.$$

Whence, solving for $v$ and $v'$,

$$2v = u(1 - e) + u'(1 + e),$$

$$2v' = u(1 + e) + u'(1 - e).$$

In particular, if $e = 1$, the bodies will exchange their velocities.

516. In Sir Isaac Newton's experiments, two equal elastic spheres were mounted as pendulums of the same length, in such a way as to be in contact when hanging, each at its lowest point. When one of the spheres so mounted is drawn aside and released, the velocity at the lowest point is readily shown to be proportional to the chord of the arc through which it has swung, and, in like manner, the chord of the arc through which it rises measures the velocity with which it leaves the lowest point. Hence, when one or both of the spheres are drawn aside and released in such a manner that impact takes place at the lowest points, the ratios of the velocities before and after impact can be determined by observation. Then every experiment gives a determination of the value of $e$. It was by experiments of this kind that Newton demonstrated the constancy of $e$ for different velocities, and determined its value for different substances. The highest value obtained by him was $e = \frac{15}{16}$ for balls of glass.

## Loss of Kinetic Energy in Impact.

517. When the bodies are inelastic, the velocities $u$ and $u'$ are reduced to the common velocity $V$, equation (3), Art. 513, and the kinetic energy after impact is

$$\tfrac{1}{2}(m + m')V^2 = \frac{(mu + m'u')^2}{2(m + m')}.$$

The total energy before impact, when reduced for comparison to the same denominator, is

$$\tfrac{1}{2}mu^2 + \tfrac{1}{2}m'u'^2 = \frac{(m^2 + mm')u^2 + (mm' + m'^2)u'^2}{2(m + m')}.$$

Subtracting from this the energy after impact, we have

$$\frac{mm'(u^2 + u'^2 - 2uu')}{2(m + m')} = \frac{mm'(u - u')^2}{2(m + m')}. \quad . \quad . \quad . \quad (1)$$

This is a positive quantity equal, in fact, to the kinetic energy of one-half the harmonic mean between the given masses moving with the velocity of impact. Thus, there is a loss of kinetic energy, as we should expect, since the impulse is due to the inertia of the bodies which resists change of velocity, and the work done by it is at the expense of the kinetic energy.

This loss of kinetic energy represents work done against molecular forces; none of it being, in this case, reconverted into kinetic energy.

518. Next let the body be elastic; then a portion of the work represented by expression (1) is, as explained in Art. 508, converted for the instant into potential energy, and the expenditure of this energy against inertia is the origin of the second impulse, $e(R)$. The difference between the total kinetic energy of the bodies with the final velocities $v$ and $v'$ and that with the common velocity $V$ is found, exactly as in the preceding article, to be

$$\frac{mm'(v - v')^2}{2(m + m')}.$$

This is a gain in kinetic energy, and, since by equation (9), Art. 514, it is $e^2$ times the loss found above, the total loss of kinetic energy is

$$\frac{mm'(1 - e^2)(u - u')^2}{2(m + m')}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is therefore the amount of energy that has permanently disappeared from its mechanical forms* in the case of elastic impact.

## Energy of Driving and of Forging.

**519.** The particular case in which the body $m'$ at rest receives a blow from the body $m$ moving with the velocity $u$, the impact being inelastic, is of frequent occurrence.

The kinetic energy of $m$ before impact, which is $\frac{1}{2}mu^2$, is called *the energy of the blow.* The velocity after impact which is common to $m$ and $m'$, as in Art. 513, is

$$V = \frac{mu}{m + m'}.$$

Hence the kinetic energy of the combined mass $m + m'$ after impact is

$$\tfrac{1}{2}(m + m')\frac{m^2u^2}{(m + m')^2} = \frac{m}{m + m'} \times \tfrac{1}{2}mu^2, \quad . \quad . \quad . \quad (1)$$

the latter form of the expression showing that it is the fractional part $\frac{m}{m + m'}$, of the energy of the blow.

**520.** This portion of the energy which remains in kinetic form immediately after impact is often at once converted into

*According to the extended doctrine of *the Conservation of Energy*, the energy which has disappeared from its mechanical forms is fully accounted for in the molecular forms of heat, sonorous vibrations, etc. Since no impact takes place without producing energy in some of these forms, no bodies are in fact perfectly elastic.

the work of driving the body $m'$ against a resistance. For example, in driving a nail of mass $m'$ into a plank, if $F$ is the mean resistance and $s$ the penetration or space through which the nail is driven, the portion of energy represented by expression (1) may be put equal to $Fs$. It will be noticed that the smaller $m'$ is relatively to $m$, the greater will be the fractional part of the energy of the blow which is thus utilized in driving $m'$.

In the case of pile-driving, the hammer of weight $W$ is raised to a height $h$ and let fall, so that the energy of the blow is $Wh$. Using this in expression (1), we have, when $F$ is the mean resistance and $s$ the penetration,

$$Fs = \frac{W^2 h}{W + W'}.$$

**521.** The remaining part of the energy of the blow, after subtracting expression (1), is

$$\frac{m'}{m + m'} \times \tfrac{1}{2} m u^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which agrees with expression (1), Art. 517, when we put $u' = 0$. This portion of the energy takes the form, as we have seen, of work done against molecular forces during the impact. It is therefore that part of the energy which is utilized in forging or producing permanent deformation of the body which receives the blow. In this case, $m'$ is not simply the mass of the piece to be forged, but includes that of the body which backs up the piece, usually taken as the anvil, so that the work done consists of the deformation of a part of the mass $m'$. Expression (2) shows that the greater $m'$ is relatively to $m$ the greater the fractional part of the energy which is utilized in forging.

The other part of the energy, which is represented by expression (1), appears immediately after the impact in the form of kinetic energy of the mass $m + m'$. The motion of this mass is, in practice, speedily checked by means of other resistances against which work is done during a brief interval subsequent to that of the original impulse.

## Oblique Impact of Spheres.

**522.** In the impact of smooth spheres moving not in the same line, the impulse takes place in the line joining their centres at the moment of impact. We shall suppose the lines of motion of the centres before impact to be in a single plane; the line of impact and the lines of motion after impact will then be in the same plane. Resolving the velocities of the masses $m$ and $m'$ along, and perpendicularly to, the line of impact, the former components are obviously related exactly as if they were the velocities in direct impact, and the latter are not affected by the impact. Taking the direction of the resolved velocity of $m$ along the line of impact as the positive direction, let $\alpha$ and $\beta$ be the angles made with it by the directions of the motion of $m$ before and after impact, and $\alpha'$, $\beta'$ the corresponding angles for $m'$. The resolved velocities of $m$ before impact along, and perpendicular to, the line of impact are $u \cos \alpha$, $u \sin \alpha$, and like expressions represent the other velocities. Hence we have

$$v \sin \beta = u \sin \alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$v' \sin \beta' = u' \sin \alpha', \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, in accordance with equations (8) and (9), Art. 514,

$$mv \cos \beta + m'v' \cos \beta' = mu \cos \alpha + m'u' \cos \alpha', \quad . \quad . \quad (3)$$

$$v \cos \beta - v' \cos \beta' = -e(u \cos \alpha - u' \cos \alpha'). \quad . \quad . \quad (4)$$

These four equations serve to determine the four resolved velocities,

$$v \sin \beta, \quad v \cos \beta, \quad v' \sin \beta' \quad \text{and} \quad v' \cos \beta',$$

when all the other quantities are known.*

* It is to be noticed that, except when the lines of motion before impact are parallel, a very slight difference in the time of passing a given point will make a great difference in the direction of the line of impact, and therefore in the values of $\alpha$ and $\alpha'$.

## The Moment of an Impulse.

523. Suppose an impulse to be applied at any point of a rigid body which is free to turn about a fixed axis. If the line of action of the impulsive force intersects the axis, an impulsive resistance is called into action (exactly as in the case of an ordinary force, Art. 25), which, acting for the same time $\tau$, produces an impulse equal and opposite to that of the impressed force. The momenta produced by the two impulses may be regarded as taking place simultaneously, so as to neutralize each other as in Art. 48, and the impulses are in statical equilibrium.

When the line of action of the impulsive force does not intersect the fixed axis, the force may be resolved, as in Fig. 72, p. 179, into two rectangular components, one parallel to the axis, and the other in a plane at right angles to it. The first is balanced by an impulsive resistance.* We need therefore consider only the impulse perpendicular to the axis.

524. Let Fig. 115 represent a section of a solid made by a plane passing through the line of action of such an impulse, the axis of rotation being perpendicular to the plane of the paper and piercing it at $G$. Rotation will now be produced by the impulse.

Let $\omega$ be the angular velocity produced, and let $GC = a$ be the perpendicular distance of the axis from the line of action of the force $P$. Then, the moment of the impulsive force is $K = aP$; and, $I$ being the moment of inertia of the solid about the given axis, equation (1), Art. 434, gives

$$aP = I\frac{d\omega}{dt}.$$

Since this is true throughout the short interval $\tau$ for which the forces act, we have, by integration (the body being assumed at rest before the impact),

$$a(P) = I\omega, \quad \ldots \ldots \ldots \quad (1)$$

* Unless the body is free to slip along, as well as to turn about, the axis; in which case, it will produce the same momentum in the direction of the axis as if the body were a particle.

in which $\omega$ is the velocity acquired in the interval. The second member of this equation is called *angular momentum*, and the equation shows that the angular momentum produced is the measure of *the moment of an impulse;* just as, in $(P) = mv$, the linear momentum produced measures the impulse directly. It enables us to measure an impulse when the only motion produced by it is one of rotation.

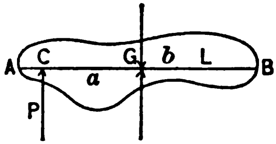


FIG. 115.

## Impulsive Pressure upon a Fixed Axis.

525. The pressure upon the axis caused by the impulse during the interval $\tau$ will consist, not only of a force equal to $P$ acting at $G$, but also of the system of forces mentioned in Art. 492, resulting from the tangential inertia of all the particles of the body.

The impulse resulting from this system of inertia forces, acting during the interval $\tau$, is found by simply replacing the common factor $\frac{d\omega}{dt}$ by $\omega$, the whole angular velocity produced by the impulse in the body supposed initially at rest. It follows that, in accordance with Art. 492, the expression for the resulting impulse is the same as that for the resultant centrifugal force in rotation about the given axis, except that $\omega$ takes the place of the factor $\omega^2$.

Hence, by Art. 486, the resultant of the system consists in general of an impulsive dyname of which the impulse is

$$(R) = \omega M\bar{r},$$

where $\bar{r}$ is the distance of the centre of inertia from the axis. Since $\omega\bar{r}$ is the velocity given to the centre of inertia, this impulse is the same in magnitude and direction as the inertia of the whole mass supposed concentrated at that point, which, as we have seen in Art. 496, is the total linear momentum communicated to the body. If the condition of Art. 488 is fulfilled, the

impulsive couple $(K)$ may be made to vanish, and the resultant reduced to a single impulse, which will not, however, generally act at the centre of inertia. (Compare Art. 488.)

526. Let us now suppose that the point $G$ in Fig. 115 is the centre of inertia, and that the axis of rotation passing through it is a principal axis, so that the line of action of the impulse lies in a principal plane. Then the impulse $(R)$, as well as the couple $(K)$, vanishes, and the impulsive pressure upon the axis now consists solely of an impulse equal to $(P)$ acting at $G$. The resistance of the axis together with the impressed impulse at $C$ forms in this case the impulsive couple whose value is $a(P)$, and the entire system of inertia impulses which resists the impressed rotation reduces to an equivalent couple whose measure is $I\omega$, the angular momentum imparted.

## Motion Produced in a Free Body by an Impulse in a Principal Plane.

527. If the body in Fig. 115, receiving an impulse $(P)$, as in the preceding article, be free, the impulse may be resolved as represented in the diagram into an equal impulse acting at $G$ and the impulsive couple $a(P)$, which produces as before the angular velocity $\omega$. The impulse acting at $G$ will now be resisted only by the linear inertia of the body, the total action of which is, by Art. 497, the same as if the whole mass were concentrated at the centre of inertia. We therefore now have the two equations

$$a(P) = I\omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$(P) = Mv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which, if $(P)$ were known, would determine the linear and angular velocities produced by the impulse.

If $k_0$ denotes the radius of gyration corresponding to the axis through $G$, we have, on eliminating $(P)$,

$$av = k_0^2\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which determines the relation between the linear and angular

velocities received. After the impulse, the body will move with these velocities unchanged, if no continuous forces act.

**528.** The motion produced is a case of plane motion (Art. 493). Let $L$, Fig. 115, be the instantaneous centre of the motion, and put $GL = b$. Then, because the body begins to move as if rotating about $L$, $v = b\omega$. Substituting in equation (3), we find

$$ab = k_0^2.$$

If the body be struck a blow at $C$, no immediate motion will be imparted to $L$, and therefore no shock will be felt at that point. It is, therefore, called *the centre of percussion* for the point $C$. The motion after the impulse will be the same as if the body were attached to a circle whose centre is $G$, and radius $b$, and this circle rolled upon a straight line passing through $L$ and parallel to the line of action of the force.

The relation between $a$ and $b$ is the same as that between $h$ and $h'$ in Art. 478, which are like $GC$ and $GL$ measured on opposite sides of $G$, $k_0$ having the same meaning as in the present case.

## Motion of a System of Bodies.

**529.** We have, in preceding chapters, considered the action of force upon a particle or a rigid body, and the motion produced, referring it to fixed points or centres of force. But, in accordance with the Third Law of Motion, no force in nature acts upon a body without an equal reaction upon some other body; in other words, all forces are of the nature of stresses between two bodies. Moreover, by the Second Law, the momenta produced by the two phases of the stress in any given time are equal and opposite. Hence, in dealing with a system of bodies, if both the bodies between which a stress acts, producing relative motion of the bodies, are included in the system, we see that the stress cannot alter the total momentum of the system. Accordingly, we have already noticed, in Art. 514, that the total momentum of two bodies is not altered by the impulses occurring in their impact.

Again, in free rotation of a rigid body, which takes place only about a principal axis passing through the centre of inertia (Art. 491), the centripetal forces, or stresses created by the centrifugal forces, while changing the momenta of separate particles, do not alter the total momentum (Art. 497).

530. *The total momentum of the system* is readily shown to be the same as *the momentum of the total mass supposed to be concentrated at, and moving with, the centre of inertia.* For, denoting the masses by $m_1, m_2, \ldots$, and referring their positions to rectangular axes, we have, for the abscissa of the centre of inertia,

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \ldots}{m_1 + m_2 + \ldots},$$

or, putting $M$ for $m_1 + m_2 + \ldots$,

$$M\bar{x} = m_1x_1 + m_2x_2 + \ldots \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating with respect to $t$,

$$M\frac{d\bar{x}}{dt} = m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} + \ldots \quad . \quad . \quad . \quad (2)$$

Hence the resolved momentum, in the direction of the axis of $x$, of $M$ at the centre of inertia is the sum of the corresponding momenta of all the particles. The same thing is true of the other resolved momenta, and therefore also of the complete momenta.

## Conservation of the Motion of the Centre of Inertia.

531. If no forces are acting the momenta of the several parts of the system are constant; hence their sum is constant, and the centre of inertia moves uniformly in a straight line. Now, since we have seen that the action of stresses between the bodies of the system does not change the total momentum, it follows that the uniform motion of the centre of inertia is not affected by the action of such stresses. This principle is known as *the conservation of the motion of the centre of inertia.*

This motion is therefore regarded as representing the motion of the system as a whole, and by *the relative motions* of the bodies of the system we understand their motions relatively to the centre of inertia. The stresses between pairs of the bodies, whether of the nature of continuous or impulsive forces, are called *the internal forces*, in distinction from *the external forces*, which have their reactions upon bodies not included in the system.

The principle may now be stated thus: The action of internal forces does not affect the motion of the system as a whole, but only the relative motion of its parts.

### The Inertia Forces of the System.

532. If we differentiate equation (2), Art. 530, we have

$$M\frac{d^2\bar{x}}{dt^2} = m_1\frac{d^2x_1}{dt^2} + m_2\frac{d^2x_2}{dt^2} + \ldots, \quad . \quad . \quad . \quad . \quad (3)$$

in which each term in the second member, being the product of a mass and its acceleration, is, when reversed, the measure of a component of its inertia. Hence, the equation expresses that the sum of the resolved forces of inertia, in the direction of the axis of $x$, is equal to the corresponding resolved inertia which the total mass would have if concentrated at and moving with the centre of inertia. Since corresponding equations exist for the other components, the forces of inertia constitute a system of which the *resultant force* is the same as that of the whole mass moving in the supposed manner. In other words, the acceleration of the centre of inertia is the same as if all the forces acting on the particles were acting with their proper magnitudes and directions upon the total mass at that point. Since the internal forces occur in pairs which neutralize each other, we may ignore them in this connection, and say that *the centre of inertia moves as if the resultant force of the system of external forces acted upon the total mass at that point.*

533. The system of forces of inertia is, by Art. 225, equivalent, not simply to the force above considered acting at $G$, the centre

of inertia, but to the dyname consisting of that force together with a couple whose axis and moment are the principal axis (Art. 227) and moment of these forces at $G$. This couple is therefore in equilibrium with the corresponding couple determined by the external forces.

When internal forces only are acting, the forces of inertia are in complete equilbrium, their moments about any axis as well as their resolved parts vanishing.

**The Hypothesis of Fixed Centres of Force.**

534. The principles explained in the preceding articles show that the *relative* motions of bodies, of which alone we can obtain any knowledge, ought to be referred to the centre of inertia of a system including all the bodies upon which the forces producing the motions react. Hence, when we use for this purpose a so-called fixed body, which in reality receives the reactions of the external impressed forces, we ignore the motions of this body caused by these reactions; in other words, we treat the fixed body as one of infinite mass. For example, in the case of falling bodies, we thus regard the earth which receives the reaction of gravity (Art. 24, and foot-note); and, in constrained motion, the body whose surface exerts the necessary force of resistance may be supposed rigidly connected with the earth considered as a body whose mass is infinite relatively to the bodies whose motion is in question.

535. So also, in treating of the orbit of the earth about the sun in Chap. X, we regarded the sun as a fixed centre of force. But in reality it is the centre of inertia of the system consisting of the earth and sun which should be taken as the fixed point of reference. Hence the symbols employed in Arts. 420–431 strictly apply to the absolute orbit described about this point, and not to the relative orbit about the sun, which is of course the orbit observed and discussed by astronomers.

Let $m$ and $M$ be the masses of the earth and sun respectively. The centre of inertia divides the distance between the earth and the sun inversely in the ratio of their masses, hence the actual distance between the bodies is not $r$ but $\rho r$, where

$$\rho = \frac{M + m}{M}.$$

Since $\rho$ is constant, the attraction $P$, which, by the law of gravitation, is proportional to the inverse square of the actual distance $\rho r$, is also proportional to the inverse square of $r$; therefore the results may all be held to apply to the orbit about the sun, that is, to the observed orbit which is similar to the actual orbit about the centre of inertia. The only difference which this change makes is in the estimate we make of the value of $\mu$, the intensity of the sun's attraction upon a unit mass.

536. Let $\mu_0$ be the true value of this intensity, and $r_0$ the actual distance between the earth and the sun. Then

$$P = \frac{\mu}{r^2} = \frac{\mu_0}{r_0^2};$$

and, since we have seen above that $r_0 = \rho r$, we find

$$\mu_0 = \rho^2 \mu.$$

Now, if $a_0$ is the mean distance of the earth from the sun, we have also $a_0 = \rho a$, and must substitute

$$a = \frac{a_0}{\rho} \quad \text{in} \quad \mu = \frac{4\pi^2 a^3}{T^2},$$

which is equation (2) of Art. 426. We have, therefore,

$$\mu_0 = \rho^2 \mu = \rho^2 \frac{4\pi^2 a_0^3}{T^2 \rho^3},$$

or

$$\mu_0 = \frac{4\pi^2 a_0^3}{T^2} \frac{M}{M + m}.$$

In order to show that the same law of attraction toward the sun governs the motions of the several planets, it was necessary for Newton to show that the quantity

$$\frac{a^3}{T^2} \frac{M}{M + m}$$

had the same value for all the planets. Since, in each case, $m$ is very small relatively to $M$, the squares of the times are very nearly proportional to the cubes of the mean distances, but not exactly, as stated in Kepler's third law.

## External and Internal Kinetic Energy of a System.

537. In separating the momentum of a system of bodies into two parts, one external and the other internal, the momentum of each particle was simply resolved into two components, one due to the motion of the centre of inertia, the other to the relative motion of the particle. It is obvious that we cannot thus simply treat the kinetic energy of the separate particles, nevertheless we shall find that the total kinetic energy of the system may be separated into parts due respectively to the motion of the centre of inertia and to the motions of the particles relatively to that point.

Let $V$ be the velocity of the centre of inertia, and let $v_1$ and $u_1$ be rectangular components of the relative velocity of $m_1$, respectively in the direction of, and in a plane perpendicular to, the line of motion of the centre of inertia. Then $u_1$ and $v_1 + V$ are rectangular components of the absolute velocity of $m_1$; hence its kinetic energy is

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_1(v_1 + V)^2,$$

or

$$\tfrac{1}{2}m_1(u_1^2 + v_1^2) + m_1v_1V + \tfrac{1}{2}m_1V^2.$$

Summing the like quantities for all the particles of the system, we have, for the total kinetic energy,

$$\tfrac{1}{2}\Sigma m(u^2 + v^2) + V\Sigma mv + \tfrac{1}{2}V^2\Sigma m.$$

Now $\Sigma mv$ (which is the projection in a given direction of the momentum of the system relative to the centre of inertia) vanishes, and $\Sigma m$ is the total mass $M$ of the system. Thus the total energy is

$$\tfrac{1}{2}\Sigma m(u^2 + v^2) + \tfrac{1}{2}MV^2,$$

of which the first term expresses the sum of the kinetic energies of the particles supposed each to have only its velocity relative to the centre of inertia, and the second term is the kinetic energy which the whole mass would have if moving with the velocity of the centre of gravity.

538. Since we have seen that the relative motions are affected by the internal forces in exactly the same way as if the centre of inertia were at rest, it follows that, when the internal forces are conservative, ***the total energy, kinetic and potential, of the system is constant,*** which proves in its greatest generality the Conservation of Energy in its mechanical forms.

### EXAMPLES. XXIV.

1. An arrow weighing 1 oz. shot from a bow, starts off with a velocity of 120 feet per second. Assuming the time of acquiring the impulse to be $\frac{1}{40}$ of a second, and $g = 32$, what is the mean value of the impulsive force in gravitation units? $9\frac{3}{8}$ pounds.

2. A ball is dropped from a height of 12 feet upon a fixed horizontal plane, and the coefficient of restitution is $\frac{4}{5}$. What height will it reach on the third rebound? 12.64 inches.

3. A sphere falls from a height $a$ above a horizontal plane and rebounds continuously. Find the whole space $s$ described, and the whole time $T$ before it is brought to rest, neglecting the time occupied by the impacts.

$$s = a\frac{1+e^2}{1-e^2};\ T = \sqrt{\frac{2a}{g}}\,\frac{1+e}{1-e}.$$

4. A ball of imperfect elasticity slides down a smooth plane of height $h$ and inclination $\alpha$. At its foot it rebounds repeatedly from a horizontal plane. Show that the ball will begin to move in a straight line when at the distance

$$\frac{2eh \sin 2\alpha}{1-e}$$

from the foot of the plane.

5. A ball weighing 10 pounds, moving with a velocity of 10 feet per second, impinges directly upon another weighing 5 pounds

which is at rest, the coefficient of restitution being $\frac{4}{5}$. What are the velocities after impact? 4 and 12 ft. per sec.

6. If in Ex. 5 the duration of the impact is $\frac{1}{100}$ of a second, find the mean value of the impulsive force in poundals, and in local pounds where $g = 32$. 6000; $187\frac{1}{2}$.

7. Two balls whose masses are as 5 : 6 impinge directly with velocities 55 and 44 feet per second in opposite directions, and $e = \frac{2}{3}$. What are the velocities after impact?

35 and 31 ft. per sec.

8. Two balls with equal velocities meet. What is the ratio of the masses if $m$ is at rest after impact? $\frac{m}{m'} = 2e + 1$.

9. Two balls with equal and opposite velocities impinge, and the first turns back with its original velocity, while the other follows with one-half that velocity. Determine the coefficient of restitution and the ratio of the masses. $e = \frac{1}{4}$; $m' = 4m$.

10. With what velocity must a ball strike an equal ball having the velocity $a$, in order to remain at rest after impact?

$$u = -a\frac{1+e}{1-e}.$$

11. A ball weighing 5 pounds moving with the velocity $7^f/_s$ is impinged upon by a ball weighing 6 pounds and moving in the same direction. If the coefficient of restitution is $\frac{5}{6}$ and the velocity of the first ball is doubled after impact, what are the velocities of the second before and after impact? $14^f/_s$; $8\frac{1}{6}{}^f/_s$.

12. A ball weighing 3 pounds moving to the right at the rate of $5^f/_s$ meets a ball weighing 4 pounds moving to the left at the rate of $1^f/_s$. The coefficient of restitution is $\frac{2}{3}$. Find the energy lost in the impact. $\frac{15}{28}$ ft.-lbs.

13. $A$, $B$ and $C$ are the masses of three bodies in a straight line, and $e$ the common coefficient of restitution. $A$ impinges on $B$ at rest, causing $B$ to impinge on $C$ at rest. Determine $B$ so that the velocity communicated to $C$ shall be a maximum.

$$B = \sqrt{AC}.$$

14. Show that the result of Ex. 13 extends to any number of masses; that is, the masses must be in geometrical progression if

the velocity of the last is to be a maximum. Show also that the velocities will be in geometrical progression, and will be equal if the common ratio of the masses is $e$.

15. A hammer weighing 2 pounds strikes a nail weighing $\frac{1}{2}$ oz. with a velocity of 30 f/s. and drives it one inch. Find the mean resistance of the wood. $F = 332.4$ lbs.

16. What must be the fall of a 4-ton hammer on a 24-ton anvil that 18 foot-tons of work may be utilized in forging?

5 ft. 3 in.

17. A body $m$ comes to rest after direct impact upon a given body $m'$ at rest. Determine $m$. $m = em'$.

18. Prove that for perfectly elastic bodies the velocities after impact are $2V - u$ and $2V - u'$, where $V$ has the same meaning as in Art. 514. Hence verify that no kinetic energy is lost in this case.

19. A body $m$ impinges obliquely upon $m'$ at rest. Show that, if $m = em'$, the bodies will move after impact in directions at right angles.

20. Two balls, weighing respectively 3 and 2 pounds, move in parallel lines whose distance is equal to half the sum of the radii. The first moves at the rate of 2 feet per second, and the other moves towards it at the rate of 3 feet per second. Supposing $e = \frac{1}{3}\sqrt{3}$, show that after impact each will move in a direction making an angle of 105° with its original direction, and with a speed equal to $\frac{1}{2}\sqrt{2}$ times its original speed.

21. Prove that if a billiard-ball strikes two adjacent cushions of imperfect elasticity, the last line of motion will be parallel to the first.

22. If the sides of the table are of lengths $a$ and $b$, and the ball starts from a point on the side $a$ at a distance $x$ from the corner, determine the angle $\alpha$ which the line of motion must make with $x$, in order that the ball may return to the point of starting, describing a parallelogram according to Ex. 21.

$$\tan \alpha = \frac{eb}{a - x(1 - e)}.$$

23. Two equal bodies $A$ and $B$ are connected by a perfectly inelastic string. $A$ slides from a pulley at the top of a smooth plane of inclination 30° and length $l$, and $B$ rests on the horizontal plane immediately under the pulley. After $A$ has slid a certain distance the string becomes taut, and $A$ then just reaches the bottom, pulling $B$ up a certain distance. Find the length of the string. $\frac{7}{8}l$.

24. Show that the loss of energy in Ex. 23 equals one-half of the kinetic energy of $A$ at the moment of the impulse, and thence derive the result found above.

25. Find the relation between the angles of incidence and reflection in the impact of an elastic particle upon a rough fixed plane, no rotation being produced.

$$e \tan \beta = \tan \alpha - \mu(1 + e).$$

26. In gun fire, regarding the projectile and gun as two rigid bodies, show that the energy of the projectile and that of recoil are inversely as the masses.

27. A uniform bar, of mass $M$ and length $2a$, can turn about a fixed axis through its centre and perpendicular to it. The bar being at rest, an inelastic ball of mass $m$ impinges upon it in a direction perpendicular to the bar and to the axis, at a point distant $c$ from the centre. Find the angular velocity after impact.

$$\omega = \frac{3mcu}{Ma^2 + 3mc^2}.$$

28. Supposing the bar in Ex. 27 to be free, find its angular velocity and its linear velocity after impact.

$$\omega = \frac{3mcu}{(M + m)a^2 + 3mc^2}; \quad v = \frac{ma^2u}{(M + m)a^2 + 3mc^2}.$$

29. Show that, if the impact in Ex. 27 is elastic, the angular velocity will be multiplied by $(1 + e)$; also find the ratio of the masses if $m$ comes to rest.

$$\frac{m}{M} = \frac{ea^2}{3c^2}.$$

30. Denoting by $k$ a principal centroidal radius of gyration of any free solid of mass $M$ impinged upon, as in Ex. 28, by a particle of mass $m$ in a line perpendicular to the principal axis and at a distance $c$ from it, find the final linear and angular velocities when the impact is elastic.

$$v = \frac{m(k^2 + c^2) - eMk^2}{Mk^2 + m(k^2 + c^2)} u,$$

$$v' = \frac{(1 + e)mk^2u}{Mk^2 + m(k^2 + c^2)},$$

$$\omega' = \frac{(1 + e)mcu}{Mk^2 + m(k^2 + c^2)}.$$

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